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A differential quadrature-based approach à la Picard for systems of partial differential equations associated with fuzzy differential equations

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ABSTRACT

Departing from a numerical method designed to solve ordinary differential equations, in this manuscript we extend such approach to solve problems involving fuzzy partial differential equations. The method proposed in this work is a non-recursive technique that combines differential quadrature rules and a Picard-like scheme in order to obtain general solutions of systems of partial differential equations derived from a general fuzzy partial differential model. The property of stability and a bound for the Hausdorff distance are established under suitable conditions. Several numerical examples are provided in order to show the effectiveness of the proposed technique.

1. Introduction

Let T be a positive real number, and let r and s be positive integers. In this manuscript, we investigate the class of fuzzy partial differential equations

$$L_t^{(r)}\tilde{u}(x,t) + \gamma L_x^{(s)}(\tilde{u}(x,t)) = \tilde{f}(x,t,\mathbf{k}), \tag{1.1}$$

subject to suitable boundary data and satisfying the initial conditions

$$L_t^{(i)}\tilde{u}(x,0) = \tilde{a}_i(x,\tilde{\mathbf{q}}), \quad \text{for all } i = 0,\ldots,r-1, \tag{1.2}$$

where $L_t^{(i)}$ is the ith-order derivative operator with respect to t for each $i = 0, 1, \ldots, r$, and $L_x^{(s)}$ the sth-order derivative with respect to x. Here, $\tilde{u}(x,t)$ represents the unknown fuzzy function on the vector $(x,t) \in [0,1] \times [0,T]$, while $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{k}}$ denote vectors consisting of fuzzy numbers.

It is worth noticing that (1.1) may be seen as the fuzzification via the Zadeh's extension principle of the same equation but without fuzzy variables and parameters. Following L.A. Zadeh [1], the introduction of 'fuzziness' in (1.1) means to take

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into account an uncertainty in the value of the equation parameters which cannot be identified as randomness. Fuzziness plays an important role in decision processes [2] and in regression analysis [3,4]. It is also an important tool in modeling realistic problems in science and engineering which traditionally involve ordinary differential equations [5,6]. In the case of partial differential equations, fuzziness may be employed also; however, the use of this tool for partial differential equations has not been investigated thoroughly in the specialized literature. Indeed, few reports have been devoted to the discussion on the fuzzification of partial differential models.

A well-known approach for obtaining analytic solutions of fuzzy partial differential equations is to fuzzify the solution of the corresponding deterministic partial differential model. This approach was originally proposed by J.J. Buckley and T. Feuring in the article [7], in which the authors discuss the existence of the so-called *Buckley–Feuring solution* through some particular boundary-value problems. It is important to mention that this method has been recently applied to non-polynomial fuzzy partial differential equations [8], and that it has been used as a way to obtain approximate solutions, too [9]. In the latter problem, the approximations have been calculated in the crisp case by means of the Adomian method, and then it was verified that they are in fact Buckley–Feuring solutions. Unfortunately, the problem with this approach is that the Buckley–Feuring solutions may not exist.

Some difference methods to find numerical solutions for linear fuzzy parabolic and hyperbolic partial differential equations have been proposed in [10]. A fuzzy two-dimensional differential transform method was used in [11] as a semi-analytical technique to calculate approximate solutions of fuzzy partial differential equations. In turn, the fuzzy solution of classical linear partial differential equations (like the heat, the wave and the Poisson equations) was obtained in [12] through the fuzzification of the deterministic solution. In the present work, we extend the approach proposed in [13] to solve (1.1). More precisely, we propose a numerical scheme combining differential quadrature rules [14] (which provide high-order finite-difference approximations) and a Picard-like recursion. In spite of its recursive nature, the proposed approach leads to a final non-recursive approximate solution by means of operational matrices and vectors of known quantities. The results obtained through this method are compared against solutions available in the literature, obtaining satisfactory results.

This work is sectioned as follows. In Section 2, we provide the basic definitions on fuzzy sets, α-cuts of fuzzy sets, fuzzy numbers, the Hausdorff distance between fuzzy numbers, Zadeh's extension principle and fuzzy partial differential equations. Section 3 is devoted to introducing the differential quadrature-based Picard-like method to solve a system of partial differential equations associated to fuzzy models of the form (1.1). In Section 4, we derive some analytical results which include a bound for the Hausdorff distance and a stability criterion. In Section 5, we provide numerical comparisons against results available in the literature, in order to illustrate the performance of our method in some known scenarios. This work closes with a section of concluding remarks.

2. Preliminaries

2.1. Definitions

We start off this section by establishing the meaning of some basic notions employed in this work. Throughout, we will use U to represent a nonempty and fixed (though arbitrary) closed interval of $\mathbb{R}$.

Definition 2.1. A **fuzzy set** $\tilde{w}$ on U is defined by a membership function $\mu_w(x) : U \to [0, 1]$. For each $x \in U$, the number $\mu_w(x)$ is called the **degree** of x in $\tilde{w}$.

From a rigorous perspective, a fuzzy set is an ordered pair (U, μ) consisting of a set U and a membership function $\mu : U \to [0, 1]$. In the practice, μ identifies the degree in which certain attribute w is present in each member of U. The notation $\tilde{w}$ serves to represent both the pair (U, μ) and the fact that the characteristic of interest is an attribute w.

Definition 2.2. Let $\tilde{w}$ be a fuzzy set on U. The crisp subset $\mathrm{supp}(\tilde{w})$ of U whose members are those elements of U which have nonzero membership grades in $\tilde{w}$ is called the **support** of $\tilde{w}$, that is, $\mathrm{supp}\,(\tilde{w}) = \{x \in U : \mu_w(x) > 0\}$.

Definition 2.3. Let $\tilde{w}$ be a fuzzy set on U, and let $\mathrm{cl}(\mathrm{supp}(\tilde{w}))$ represent the closure of the support of $\tilde{w}$ in the topology of U relative to the standard topology of $\mathbb{R}$. An α-**cut** of $\tilde{w}$ is the crisp set denoted by $[\tilde{w}]_\alpha$, and defined by

$$[\tilde{w}]_\alpha = \begin{cases} \{x \in U : \mu_w(x) \geq \alpha\}, & \text{if } \alpha > 0, \\ \mathrm{cl}(\mathrm{supp}(\tilde{w})), & \text{if } \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.4. A fuzzy set $\tilde{w}$ on U is a **fuzzy number** if the following conditions are satisfied:

- $\tilde{w}$ is **normal** on U, that is, $\max_{x \in U} \mu_w(x) = 1$,
- $\tilde{w}$ is **convex** on U, meaning that the inequality $\mu_w(\lambda x + (1 - \lambda)z) \geq \min(\mu_w(x), \mu_w(z))$ is satisfied for each $x, z \in U$ and each $\lambda \in [0, 1]$,
- $\tilde{w}$ is upper semi-continuous and
- $\tilde{w}$ has bounded support.

In view that α-cuts of fuzzy numbers are always closed and bounded intervals [15], we can readily rewrite them in parametric form. The next definition describes such representation.