Journal of Computational and Applied Mathematics 299 (2016) 68–91

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Discontinuous Galerkin finite element differential calculus and applications to numerical solutions of linear and nonlinear partial differential equations

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ARTICLE INFO

Article history:
Received 31 December 2014
Received in revised form 16 October 2015

MSC:
65D25
65N06
65N12
65N30

Keywords:
Weak derivatives
Numerical derivatives
Differential calculus
Discontinuous Galerkin finite elements
Linear and nonlinear PDEs
Numerical solutions

ABSTRACT

This paper develops a discontinuous Galerkin (DG) finite element differential calculus theory for approximating weak derivatives of Sobolev functions and piecewise Sobolev functions. By introducing numerical one-sided derivatives as building blocks, various first and second order numerical operators such as the gradient, divergence, Hessian, and Laplacian operator are defined, and their corresponding calculus rules are established. Among the calculus rules are product and chain rules, integration by parts formulas and the divergence theorem. Approximation properties and the relationship between the proposed DG finite element numerical derivatives and some well-known finite difference numerical derivative formulas on Cartesian grids are also established. Besides independent interest in numerical differentiation, the primary motivation and goal of developing the DG finite element differential calculus is to solve partial differential equations. It is shown that several existing finite element, finite difference and discontinuous Galerkin methods can be rewritten compactly using the proposed DG finite element differential calculus framework. Moreover, new discontinuous Galerkin methods for linear and nonlinear PDEs are also obtained from the framework.

1. Introduction

Numerical differentiation is an old but basic topic in numerical mathematics. Compared to the large amount of literature on numerical integration, numerical differentiation is a much less studied topic. Given a differentiable function, the available numerical methods for computing its derivatives are indeed very limited. There are essentially only two such methods (cf. [1]). One method is to approximate derivatives by difference quotients. The other is to first approximate the given function (or its values at a set of points) by a simpler function (e.g., polynomial, rational function and piecewise polynomial) and then to use the derivative of the approximate function as an approximation to the sought-after derivative. The two types of classical methods work well if the given function is sufficiently smooth. However, the two classical methods produce large errors or divergent approximations if the given function is rough, which is often the case when the function is a solution of a linear or nonlinear partial differential equation (PDE).

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http://dx.doi.org/10.1016/j.cam.2015.10.024
0377-0427/

For boundary value and initial–boundary value problems, classical solutions often do not exist. Consequently, one has to deal with generalized or weak solutions which are defined using a variational setting for linear and quasilinear PDEs. Although numerical methods for PDEs implicitly give rise to methods for approximating weak derivatives (in fact, combinations of weak derivatives) of the solution functions (cf. [2–5]), to the best of our knowledge, there is no systematic study and theory in the literature on how to approximate weak derivatives of a given (not-so-smooth) function. Moreover, for linear second order PDEs of non-divergence form and fully nonlinear PDEs, it is not possible to derive variational weak formulations using integration by parts. As a result, weak solution concepts for those types of PDEs are different. The best known and most successful one is the *viscosity solution* concept (cf. [6,7] and the references therein). To directly approximate viscosity solutions, which in general are only continuous functions, one must approximate their derivatives in some appropriately defined sense offline (cf. [8,9]) and then substitute the numerical derivatives for the (formal) derivatives appearing in the PDEs. Clearly, to make such an intuitive approach work, the key is to construct "correct" numerical derivatives and to use them judiciously to build numerical schemes.

This paper addresses the above two fundamental issues. The specific goals of this paper are twofold. *First*, we systematically develop a computational framework for approximating weak derivatives and a new discontinuous Galerkin (DG) finite element differential calculus theory. Keeping in mind the approximation of fully nonlinear PDEs, we introduce locally defined, one-sided numerical derivatives for piecewise weakly differentiable functions. Using the newly defined one-sided numerical derivatives as building blocks, we then define a host of first and second order sided numerical differential operators including the gradient, divergence, curl, Hessian and Laplace operators. To ensure the usefulness and consistency of these numerical operators, we establish basic calculus rules for them. Among the rules are the product and the chain rule, integration by parts formulas and the divergence theorem. We establish some approximation properties of the proposed finite element numerical derivatives and show that they coincide with well-known finite difference derivative formulas on Cartesian grids. Consequently, our finite element numerical derivatives are natural generalizations of well-known finite difference numerical derivatives on general meshes. These results are of independent interest in numerical differentiation. *Second*, we present some applications of the proposed DG finite element differential calculus to build numerical methods for linear and nonlinear partial differential equations. This is done based on a very simple idea; that is, we replace the (formal) differential operators in the given PDE by their corresponding DG finite element numerical operators and project (in the L^2 sense) the resulting equation onto the discontinuous Galerkin finite element space V_r^h. In addition, we include additional stability terms if necessary. We show that the resulting numerical methods not only recover several existing finite difference, finite element and discontinuous Galerkin methods, but also give rise to some new numerical schemes for both linear and nonlinear PDE problems.

The remainder of this paper is organized as follows. In Section 2 we introduce the mesh and space notation used throughout the paper. In Section 3 we give the definitions of our DG finite element numerical derivatives and various first and second order numerical differential operators. In Section 4 we establish an approximation property and various calculus rules for the DG finite element numerical derivatives and operators. In Section 5 we discuss the implementation aspects of the numerical derivatives and operators. Finally, in Section 6 we present several applications of the proposed discontinuous Galerkin finite element differential calculus to numerical solutions of prototypical linear and nonlinear PDEs including the Poisson equation, the biharmonic equation, second order linear elliptic PDEs in non-divergence form, first order fully nonlinear Hamilton–Jacobi equations and second order fully nonlinear Monge–Ampère equations.

2. Preliminaries

Let d be a positive integer, $\Omega \subset \mathbf{R}^d$ be a bounded open domain, and $\mathcal{T}_h$ denote a locally quasi-uniform and shape-regular partition of Ω [10]. Let $\mathcal{E}_h^I$ denote the set of all interior faces/edges of $\mathcal{T}_h$, $\mathcal{E}_h^B$ denote the set of all boundary faces/edges of $\mathcal{T}_h$, and $\mathcal{E}_h := \mathcal{E}_h^I \cup \mathcal{E}_h^B$.

Let $p \in [1, \infty]$ and $m \geq 0$ be an integer. Define the following piecewise $W^{m,p}$ and piecewise C^m spaces with respect to the mesh $\mathcal{T}_h$:

$$W^{m,p}(\mathcal{T}_h) := \prod_{K\in\mathcal{T}_h} W^{m,p}(K), \qquad C^m(\mathcal{T}_h) := \prod_{K\in\mathcal{T}_h} C^m(\overline{K}).$$

When $p = 2$, we set $H^m(\mathcal{T}_h) := W^{m,2}(\mathcal{T}_h)$. We also define the analogous piecewise vector-valued spaces as $\boldsymbol{H}^m(\mathcal{T}_h) := [H^m(\mathcal{T}_h)]^d$, $\boldsymbol{W}^{m,p}(\mathcal{T}_h) = [W^{m,p}(\mathcal{T}_h)]^d$, $\boldsymbol{C}^m(\mathcal{T}_h) = [C^m(\mathcal{T}_h)]^d$, and the matrix-valued spaces $\tilde{\boldsymbol{H}}^m(\mathcal{T}_h) := [H^m(\mathcal{T}_h)]^{d\times d}$, $\tilde{\boldsymbol{W}}^{m,p}(\mathcal{T}_h) := [W^{m,p}(\mathcal{T}_h)]^{d\times d}$, and $\tilde{\boldsymbol{C}}^m(\mathcal{T}_h) = [C^m(\mathcal{T}_h)]^{d\times d}$. The piecewise L^2-inner product over the mesh $\mathcal{T}_h$ is given by

$$(v, w)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K vw\,dx,$$

and for a set $\mathcal{S}_h \subset \mathcal{E}_h$, the piecewise L^2-inner product over $\mathcal{S}_h$ is given by

$$\langle v, w\rangle_{\mathcal{S}_h} := \sum_{e\in\mathcal{S}_h} \int_e vw\,ds.$$

Angled brackets without subscripts $\langle\cdot,\cdot\rangle$ represent the dual pairing between some Banach space and its dual.