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Local error estimates for least-squares finite element methods for first-order system

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ABSTRACT

We present local energy type error estimates for first-order system div least-squares (LS) finite element methods. The estimate shows that the local energy norm error is bounded by the local best approximation and weaker norms which account for the pollution. The estimate is similar to the one for the standard Galerkin methods. However, our estimate needs to consider the effect of error of dual (flux) variables since LS methods approximate the primary and dual variables simultaneously. The effect of error of the dual variables is shown to be of higher order. Moreover, our estimate shows the convergence behavior when locally enriched approximation spaces are used in the area of interest. As an elementary consequence of the estimate, asymptotically exact a posteriori error estimator is constructed for the local area of interest under mild assumptions.

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1. Introduction

Least-squares (LS) finite element methods based on a system of first-order are successfully used to approximate solutions of partial differential equations (PDEs) arising from sciences and engineering. The methods approximate the primary and dual (flux) variable simultaneously, while the inf–sup condition is not required for the approximation spaces, see [1,2]. In other words, the approximation spaces for the primary and dual variables can be chosen independently. Also, the resulting algebraic system is symmetric and positive definite. These are considered as some of the main advantages of the LS methods, see [3–5].

For the application of the LS methods, many considered that a weakness of the LS methods is the requirement of high regularity solutions for the underlying partial differential equations. This is mainly due to that fact that the basic error estimates indeed require high regularity solutions for optimal rate of convergence in the natural energy norm. However, various error estimates are developed for the methods, which show optimal rate of convergence without assuming high regularity solutions [6–9]. In particular, L_2 norm error estimates are obtained for the primary and dual variables separately, and the estimates show optimal rate of convergence with minimum regularity assumption, i.e. H^{1+s} with $1/2 < s \leq 1$, see [9]. This opens the door for an application of LS methods for wide variety of problems.

While there has been a great progress in error estimates for the LS methods, there are very limited local error estimates for the methods. Moreover, most of the local error estimates rely on supercloseness property of LS solutions in global norms. For example, the pointwise error estimates obtained in [10] essentially used supercloseness of LS solutions and the standard Galerkin solutions. It requires global smoothness of the underlying solutions of PDEs. Recently, the pointwise error estimates for LS solutions without relying the supercloseness is presented in [11]. The LS solutions is considered as a solution of perturbation form of the Galerkin methods and the results concerning the perturbation form in [12] is used.

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In this paper, we present local energy type error estimates for the LS methods. The same type of error estimates are developed for the standard Galerkin methods [13]. As pointed out in [14], *in many practical problems, the phenomenon of interest is much smaller than the typical problem size, and it is critical to concentrate the computational effort to guarantee local accuracy in a preassigned region*. Hence, it is imperative to obtain local error estimates for LS approach. Since LS methods approximate the primary and dual variables simultaneously, we need to consider the effect of dual approximation on the primary variable. Our estimates show that LS solutions are locally best approximation with higher order pollution terms. In particular, the effect of dual variables on the primary variable is weak and higher order without any additional assumption on the global regularity of solutions. Unlike the energy norm error estimates for the Galerkin methods, we do not consider local approximation solutions and we do not rely on discrete harmonic functions, see [13]. Our estimates rely on the superapproximation properties and orthogonality of LS solutions.

As an application of the local energy error estimates, we present a posteriori error estimates for the LS methods. More precisely, we construct asymptotically exact a posteriori error estimators for the methods for a preassigned area of interest. We use the idea from [15]. Due to the local error estimates developed in this paper, an efficient way of developing asymptotically exact error estimators is presented. As mentioned before, LS methods allow to choose the approximation spaces without satisfying inf–sup condition. We take advantage of this by using the approximation space for the primary function u as locally enriched polynomial spaces. With the asymptotically exact estimators developed in this paper along with error indicators in [16], one can develop efficient adaptive procedures for LS methods. This is a topic of future study.

The paper is organized as follows. Section 2 introduces mathematical equations for the second-order scalar elliptic partial differential equations and the div least-squares formulation is described for the equations. In Section 3, finite element spaces are introduced and the least-squares approximate solutions are defined. In Section 4, we establish local energy type error estimate for the primary variable. In Section 5, we construct asymptotically exact a posteriori error estimators for the area of interest under mild assumptions.

1.1. Notation

Let $H^s(\Omega)$ denote the Sobolev space of order s defined on Ω. The norm in $H^s(\Omega)$ will be denoted by $\|\cdot\|_s$. For $s = 0$, $H^s(\Omega)$ coincides with $L_2(\Omega)$. We shall use the spaces

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\},$$
$$\mathbf{W} = \{\boldsymbol{\sigma} \in (L^2(\Omega))^n : \nabla \cdot \boldsymbol{\sigma} \in L^2(\Omega) \text{ and } \mathbf{n} \cdot \boldsymbol{\sigma} = 0 \text{ on } \Gamma_N\},$$

with norms

$$\|u\|_1^2 = (u, u) + (\nabla u,\ \nabla u),$$
$$\|\boldsymbol{\sigma}\|_{H(\mathrm{div})}^2 = (\nabla \cdot \boldsymbol{\sigma},\ \nabla \cdot \boldsymbol{\sigma}) + (\boldsymbol{\sigma},\ \boldsymbol{\sigma}).$$

For $D \subset \Omega$ and $k = 0, 1, 2$,

$$\|v\|_{k,D} = \left(\int_D \sum_{|\alpha| \le k} (D^\alpha v(x))^2 dx \right)^{1/2}.$$

2. Problem formulation

Let Ω be a polyhedral domain in $\mathbb{R}^n$, $n = 2, 3$, with Lipschitz boundary $\partial\Omega = \Gamma_D \bigcup \Gamma_N$. We assume that Γ_D is nonempty. Let $\mathbf{n}$ denote the unit outward normal vector to the boundary. We consider a system of first-order equations:

$$\begin{aligned} &\boldsymbol{\sigma} + \mathcal{A}\nabla u = 0 \quad \text{in } \Omega, \\ &\nabla \cdot \boldsymbol{\sigma} + c\, u = f \quad \text{in } \Omega, \end{aligned} \tag{2.1}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma} = 0 \quad \text{on } \Gamma_N \tag{2.2}$$

where the symbols $\nabla\cdot$ and ∇ stand for the divergence and gradient operators, respectively; A is a given $n \times n$ uniformly bounded and symmetric positive definite matrix; and f is a given scalar function. We assume that there exists a unique solution to (2.1).

Note that the above system of equations can be transformed into the following second-order equation:

$$-\nabla \cdot \mathcal{A}\nabla u + cu = f \quad \text{in } \Omega, \tag{2.3}$$

with the same boundary conditions.

We assume the following *a priori* estimate for u satisfying (2.1): there exists a positive constant C independent of f satisfying

$$\|u\|_{1+s} \le C\, \|f\|_{-1+s} \le C\, \|f\|_0 \quad \text{for } \frac{1}{2} < s \le 1. \tag{2.4}$$