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A continuing exploration of a decomposed compact method for highly oscillatory wave problems

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ABSTRACT

This paper concerns a highly effective and decomposed compact scheme for solving a highly oscillatory paraxial Helmholtz problem in radially symmetric fields. The decomposition is utilized in the transverse direction to eliminate the singularity of the differential equation in polar coordinates. Numerical stability of the splitting scheme is investigated. It is shown that the numerical method introduced is not only highly accurate and efficient due to its straightforward algorithmic structure, but also stable under reasonable constraints for practical applications. Computational examples are presented to illustrate our conclusions.

1. Introduction

As seen in several recent publications [1–5], together with Lorentz force law, Maxwell's equations have been playing a fundamental role in modern optics, electrodynamics, and electrical circuit designs. These principal fields in turn provide the ultimate theoretical backbone for many cutting-edge technologies.

Maxwell's field equations are coupled first-order differential equations which describe how electric and magnetic fields propagate and interact. These coupled equations are not well suited for use in typical initial–boundary value problem computations. However, when these equations are decoupled, we acquire the following time-dependent Helmholtz equation [1–3]:

$$u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{\zeta\zeta}\right), \quad (x, y) \in \mathcal{D}_2,\ \zeta > \zeta_0,\ t > t_0, \tag{1.1}$$

where $u = u(x, y, \zeta, t)$ is the intensity function of the electric field, $\mathcal{D}_2$ is the two-dimensional transverse domain, ζ is the beam propagation direction and c is the speed of light. In the circumstance when a monochromatic beam is concerned, we

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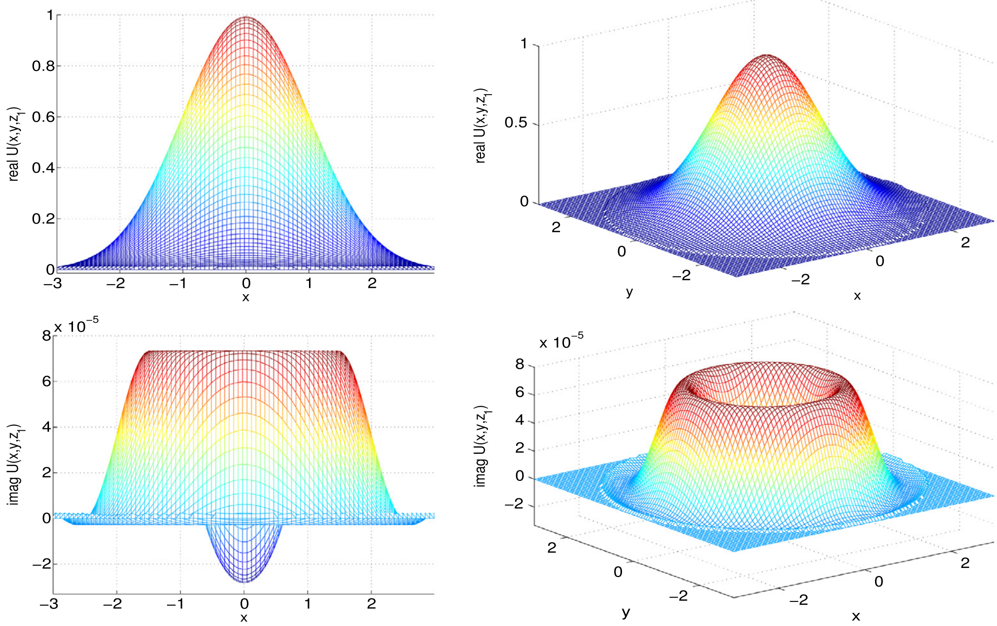

Fig. 1.1. An x-projection (left) and three-dimensional view (right) of the real and imaginary parts of a typical complex amplitude $U(x, y, z)$ at focusing point z_1, respectively.

may denote $u(x, y, \zeta, t) = U(x, y, \zeta)e^{2\pi i\nu t}$, where $i = \sqrt{-1}$, ν is the frequency of the optical wave and U is the complex wavefunction [1,6]. Substituting this into (1.1), we arrive at

$$U_{xx} + U_{yy} + U_{\zeta\zeta} = -\kappa^2 U, \quad (x, y) \in \mathcal{D}_2,\ \zeta > \zeta_0, \tag{1.2}$$

where $\kappa = 2\pi\nu/c > 10^2$ is the wave number of the light.

Let $E(x, y, \zeta) = U(x, y, \zeta)e^{i\kappa\zeta}$ be the complex envelope of U. Thus, from (1.2) we observe that

$$2i\kappa E_\zeta = E_{xx} + E_{yy} + E_{\zeta\zeta}, \quad (x, y) \in \mathcal{D}_2,\ \zeta > \zeta_0.$$

In Fig. 1.1, we show a typical complex amplitude function U of the solution of (1.2) at z_1 under suitable initial and boundary conditions, where z_1 is the focusing position of the optical wave. While in Fig. 1.2, we demonstrate its corresponding complex envelope function E [5]. In each figure the parameter $\kappa\zeta = 0.001$ is utilized. It can be observed that the shapes of the functions are well preserved throughout variable transformations.

Because of the slow change of E in transverse directions, we may further assume that [2–4]

$$E_{\zeta\zeta} \approx \kappa^2 pE,$$

where p is a constant refractive parameter of the light system. This leads to the paraxial Helmholtz equation,

$$2i\kappa E_\zeta = E_{xx} + E_{yy} + \kappa^2 pE, \quad (x, y) \in \mathcal{D}_2,\ \zeta > \zeta_0, \tag{1.3}$$

which describes the propagation of electromagnetic waves in the form of either paraboloidal waves or Gaussian beams. Because most lasers emit beams of this form, this equation is used frequently in beam propagation computations in particular within focal regions [6,7].

Although investigations of numerical solution procedures of (1.1)–(1.3) along with different boundary and initial data can be found in numerous recent publications, the study of highly efficient numerical algorithms for the highly oscillatory propagation solutions and beam–material interactions is still in its infancy [8]. In this paper, we are particularly interested in decomposition schemes that are highly accurate in transverse directions as well as highly efficient and effective for practical applications. We focus our attention on cases where radially symmetric electric fields in transverse directions are anticipated. The singularity emerging in the subsequent paraxial Helmholtz equation in polar coordinates is removed successively via a decomposition strategy [9,10]. Special attention is paid to the numerical stability of the decomposition algorithm proposed.

Our discussion is organized as follows. In Section 2, the paraxial differential equation is decomposed based on separation of the transverse domain. The singularity of the polar differential equation is removed, and the accuracy of approximation is retained via an asymptotic expansion. A tridiagonal linear system is acquired once a full discretization is accomplished. Section 3 is devoted to detailed analysis of the numerical stability. It is shown that the numerical algorithm constructed is stable if reasonable constraints are satisfied. Remarks are given if variations of the algorithmic settings are introduced. Section 4 presents simulation experiments of typical highly oscillatory paraxial optical wave problems. Satisfactory beam propagation computations and focusing phenomena are demonstrated. Finally, in Section 5, concluding remarks are given.