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High order finite volume WENO schemes for the shallow water flows through channels with irregular geometry

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ABSTRACT

The shallow water equations are widely used to model flows in rivers and coastal areas. In this paper, we consider the shallow water flows in open channels with irregular geometry and a non-flat bottom topography, and design high order finite volume weighted essentially non-oscillatory (WENO) methods. A special source term approximation is introduced so that the proposed methods can preserve the still water steady state exactly. We also employ a simple positivity-preserving limiter to provide efficient and robust simulations near the wetting and drying front. The proposed methods are well-balanced for the still water steady state solutions, preserve the non-negativity of the wet cross section, and are genuinely high order accurate in smooth regions for general solutions and essentially non-oscillatory for general solutions with discontinuities. Numerical examples are performed at the end to verify these properties.

1. Introduction

The shallow water equations are widely used in the modeling and simulation of free surface flows in rivers and coastal areas, and can predict tides, storm surge levels and coastline changes from hurricanes and ocean currents. In this paper, we consider shallow water flows in open channels with irregular geometry and a non-flat bottom topography, which take the form

$$\begin{aligned} &H_t + Q_x = 0 \\ &Q_t + \left(\frac{Q^2}{H} + \frac{1}{2}g\sigma h^2\right)_x = \frac{1}{2}gh^2\sigma_x - g\sigma h b_x \end{aligned} \tag{1.1}$$

where h denotes the water height, b represents the bottom topography, σ is the breadth of the rectangular channel, $H = \sigma h$ is the wet cross section, $Q = Hu$ is the mass flow rate, u is the velocity, and g is the gravitational constant. The source term $-g\sigma h b_x$ accounts for the effect of non-flat bottom topography, and the other source term $gh^2\sigma_x/2$ comes from the variation of the cross section. Other source terms, such as a friction term, could also be added. When the cross section $\sigma(x)$ is a constant, this model reduces to the shallow water equations with a non-flat bottom topography.

For the shallow water equations and other conservation laws with source terms, one main difficulty in solving them numerically is the treatment of source terms, which need to be balanced by the flux gradients at the steady state. Otherwise, these methods may introduce spurious oscillations near the steady state, making it challenging to simulate small perturbations of such state. Well-balanced schemes are specially designed to preserve exactly these steady state solutions up to machine error with relatively coarse meshes, and have been an active research area in the past two decades. Many

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researchers have developed well-balanced methods for the shallow water equations using different approaches. We refer the readers to [1–6] and the references therein. Another challenge encountered in the simulations of the shallow water model is the appearance of dry or near-dry areas, where no or little water is present. Numerically, negative water height may be produced if no special attention is paid in such area, which may cause the computation to fail as the system loses its hyperbolicity. Various positivity-preserving techniques have been studied to overcome this difficulty, and we refer to [7–11] for some recent related work.

Most of the above work is for the shallow water equations. For the shallow water flow in open channels (1.1), less work can be found in the literature. Vázquez–Cendón [12] presented a well-balanced method by rewriting the model in an equivalent form with computational variables (h, hu) and two additional source terms which account for the variable cross section $\sigma(x)$ and are zero at the steady state. Any well-balanced method for the shallow water equations can be extended here directly. Well-balanced methods based on extensions of Roe's discretization with proper flux difference splitting were given in [13]. Balbás and Karni [14] presented second-order well-balanced positivity-preserving numerical methods for the shallow water flow in rectangular channels, extending the results for the shallow water equations in [9]. Later, Hernández-Duenas and Karni [15] extended their results to the shallow water flow with arbitrary cross section, and designed well-balanced Roe-type upwind methods. Murillo and García-Navarro [16] recently proposed well-balanced method based on energy balanced arguments.

High-order accurate numerical schemes (with higher than second-order accuracy) have attracted increasing attention in many computational fields. Examples include finite difference/volume weighted essentially non-oscillatory (WENO) schemes, spectral methods and discontinuous Galerkin (DG) methods. They have been applied to solve the shallow water equations, and some of them are well-balanced and positivity-preserving, but we do not see such methods for the shallow water flow in channels. The main objective of this paper is to develop high-order finite volume WENO methods for the shallow water flows (1.1) in open channels with rectangular cross section. The proposed methods are genuinely high-order, well-balanced for the steady state solution and preserve the non-negativity of the wet cross section without loss of mass conservation.

This paper consists of four additional sections. In Section 2, the mathematical model and its steady state solutions are described. The well-balanced algorithm is presented in Section 3. We propose a novel source term approximation, which is not only high order accurate, but also well-balanced. Coupled with well-balanced numerical fluxes, the resulting WENO methods are shown to capture the steady state solution exactly. In Section 4, we demonstrate that the first order version of the proposed methods preserves the non-negativity of water height, and then show that, high order WENO methods, coupled with a simple positivity-preserving limiter, maintain this property. The positivity-preserving limiter keeps the water height non-negative, preserves the mass conservation and at the same time does not affect the high-order accuracy for the general solutions. Finally, in Section 5 we provide some numerical experiments to gauge the performance of the proposed well-balanced positivity-preserving WENO methods for the shallow water model in open channels, demonstrating the accuracy and robustness of the proposed methods for a wide range of shallow water flows.

2. The shallow water model in open channels

As simplified models of some free surface flows, the shallow water equations for flows in an open channel with variable cross section take the form

$$\begin{aligned} &H_t + Q_x = 0 \\ &Q_t + \left(\frac{Q^2}{H} + I_1\right)_x = I_2 - g\sigma_b h b_x, \end{aligned} \tag{2.1}$$

where $\sigma^0(x, z)$ is the breadth of the channel, $\sigma_b(x) = \sigma^0(x, b(x))$ is the bottom channel width, $H = \int_b^{h+b} \sigma^0(x, z)dz$ is the cross-sectional wet area, and $Q = Hu$ is the mass flow rate. I_1 is given by $I_1 = g\int_h^{h+b}(h + b - z)\sigma^0(x, z)dz$ which is equal to the cross-sectional average of the hydrostatic pressure multiplied by H, and $I_2 = g\int_h^{h+b}(h + b - z)\sigma_x^0(x, z)dz$.

The Eqs. (2.1) have the hydrostatic pressure that cannot be directly expressed in terms of the computational variables (H, Q). By some simple algebra, one can show that it is equivalent to the following non-conservative formation:

$$\begin{aligned} &H_t + Q_x = 0 \\ &Q_t + \left(\frac{Q^2}{H}\right)_x + \frac{g}{2\sigma_t}(H^2)_x = \frac{gH}{\sigma_t}(I_3 - \sigma_b b_x), \end{aligned} \tag{2.2}$$

which can be further written in the matrix form:

$$\begin{pmatrix} H \\ Q \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ u^2 - c_0^2 & 2u \end{pmatrix}\begin{pmatrix} H \\ Q \end{pmatrix}_x = \begin{pmatrix} 0 \\ c^2\,(I_3 - \sigma_b b_x) \end{pmatrix}, \tag{2.3}$$

where $\sigma_t(x) = \sigma^0(x, h(x) + b(x))$, $c_0^2 = gH/\sigma_t$, and $I_3 = \int_b^{h+b}\sigma_x(x, z)dz$. Numerical methods for the non-conservative hyperbolic system remain a difficult task, although there have been some recent developments along this direction [17,18].