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A geometric multigrid solver for the free-surface equation in environmental models featuring irregular coastlines

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A B S T R A C T

A recently developed multigrid method (Botto, 2013), based on the concept of volume fraction, has been tested for the inversion of the Helmholz-type equation for the free surface in the environmental public domain code COHERENS. The volume fraction concept is particularly interesting for coarse grid cells that are agglomerated from both dry and wet fine grid cells at irregular coastlines. At these locations, modifying the prolongation operator and the coarse grid discretization, using the volume fraction, results in better convergence. However, as convergence deteriorates in the case of small, elongated islands that tend to disappear by the multigrid coarsening procedure, a correction is proposed, yielding good convergence rates, irrespective of the presence of small or large islands. The method is tested extensively for the inversion of the academic Poisson equation. Larger test cases, solving the Helmholz-type equation, prove the applicability for real-life applications of environmental flows.

1. Introduction

When performing numerical simulations, in many applications an elliptic Poisson-like equation arises, which has to be inverted in an efficient way. Examples include the pressure Poisson equation in incompressible fluid flow and the implicit treatment of diffusion terms in mass and heat transfer applications. In environmental fluid mechanics, such an elliptic equation arises whenever the water level (or free surface) is treated implicitly and a time dependent Poisson equation (the Helmholz equation), describing the water wave dynamics, is obtained [1]. The inversion of the elliptic equation forms the most time-consuming part of simulation codes and hence requires an efficient numerical solver.

The efficient inversion of the Poisson equation has historically been dealt with by many researchers. As standard iterative solvers (Jacobi, Gauss–Seidel) feature poor convergence characteristics, more efficient solvers have been developed, including Krylov-type solvers, such as preconditioned conjugate-gradient (PCG) and geometric and algebraic multigrid solvers (GMG and AMG). From the available solutions, it is generally believed that geometric multigrid solvers have the greatest potential in inverting the Poisson equation in the most efficient way, both in terms of memory requirements and CPU-time, particularly for rectangular, uniform and isotropic domains. The theory and applications concerning GMG are well

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established (e.g. [2]), however, there remain certain cases where the convergence deteriorates or even fails, which form the subject of current research activities in this field.

One of these issues is the treatment of boundaries that do not align with the grid on all levels of coarsening. Indeed: an essential ingredient of GMG is the representation of the discretized equation on recursively coarsened meshes. This strategy follows from the observation that standard iterative solvers tend to resolve the high-frequency component of the error rather quickly, but fail to do so for the low-frequency components. On coarser meshes, this low-frequency component is seen as high-frequent relative to the grid spacing. By using cheap standard iterative solvers on different levels of coarsened grids, all components of the error can be resolved at high convergence rates. In GMG, a grid cell of a coarser level is obtained by agglomeration of a number of fine grid cells. For structured grids, a coarse grid cell typically consists of 4 fine grid cells in 2-D. Wherever boundaries intersect these cells, special measures are needed. Such cases frequently appear in case of immersed boundaries, but also if a stair-case or voxelized representation of the boundary is present on the finest grid level, coarser cells may consist of a combination of interior and exterior cells.

The problem is even more troublesome in case of small islands, that have a tendency to disappear at coarse levels. In that case, the geometric discrepancy between the fine and coarser levels introduces spurious eigenmodes, that are not damped effectively by the algorithm. Mikulinsky [3] addresses the issue by defining a region of influence. On this basis, it is determined whether the island has to disappear on coarser levels or not. Good convergence rates are achieved by additional smoothing operations in a region near the boundary or by a recombination of iterants [4]. Because of the disappearance of small geometric features, the latter approach was also found necessary in [5] where geometric multigrid was used as a preconditioner for the conjugate-gradient solution of the voxelized Poisson equation. The multigrid solution of [6] requires additional relaxation steps near boundaries, where the interpolation of inner cells is treated separately from the ghost cells, thereby extrapolating the values at cells inside the solid domain. A special type of boundary condition enforcement is introduced, making use of dual time stepping to stabilize the solution. In [7], the efficiency of GMG, AMG and PCG is compared for a (Dirichlet) boundary, prescribed by the level-set method, favoring the GMG approach, provided the interpolation near boundaries is adjusted to properly take into account the boundary position. In combination with the immersed boundary method, Zhu and Peskin [8] use GMG in combination with a spreading of the boundary over a number of grid cells at coarser levels, whereas Udaykumar et al. [9] introduce the concept of *volume fraction* to determine whether a coarsened cell, agglomerating fluid and solid fine cells, is treated as fluid or solid. [10] addresses the *volume fraction* as a *mask function* to determine the position of the boundary, which is used to adjust the stencil of the coarse grid discretization. The technique is referred to as second order accurate, but fails to converge in case of geometric discrepancies between grid levels. In this case, a first order reconstruction of the boundary is used instead.

In the present paper, we follow the strategy of Botto [11], showing many similarities with [9]. In [11], the concept of volume fraction is further elaborated and good convergence rates are obtained for relatively large solid inclusions. His GMG implementation has the benefit of being straightforward to implement, requiring only small modifications to the standard GMG solver for domains without inclusions or irregular boundaries. We implemented the solver in the open-domain software COHERENS [12] for 3D hydrostatic free surface flow, to efficiently solve the Helmholz equation for the free surface. Focusing first on the inversion of the Poisson equation, we show good convergence characteristics for test cases with large geometric features (i.c. irregular coastlines). In particular, the test cases, presented in [11] are successfully reproduced. However, for geometric features that tend to disappear (i.c. narrow, elongated islands), convergence is observed to deteriorate. Inspired by [7,10], we present a simple GMG strategy for the combined presence of large and small geometric features, that shows good convergence rates, irrespective of the presence of these features.

The paper is organized as follows. In Section 2, the governing equations and discretizations are presented. Section 3 provides the details of the GMG algorithm and the different measures implemented to deal with the large and small geometric boundaries. Section 4 shows the numerical results for a set of academic test cases. Finally, Section 5 illustrates the applicability of the method on a real-life application.

2. Governing equations and discretization

The multigrid solver is implemented in COHERENS [12] and is used to invert the elliptic equation for the free surface that arises from the semi-implicit solution algorithm, implemented previously in the version v2.4 [1]. COHERENS is a 3D hydrostatic code, that can be used to simulate tidal flows, in combination with transport of salinity, temperature, sediment or biological components. The motion of the flow is governed by the Navier–Stokes, equations, in which the pressure is treated as purely hydrostatic. This simplification is interesting, because it eliminates the inversion of the pressure Poisson equation. Instead, a 2D, integrated, elliptic equation for the free surface needs inversion. The governing equations are in 3D:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} = \mathrm{Adv}_x\,(u, v, w) + \mathrm{Diff}_x\,(u, v, w) + \mathrm{S}_x\,() - g\frac{\partial \zeta}{\partial x} \tag{2}$$

$$\frac{\partial v}{\partial t} = \mathrm{Adv}_y\,(u, v, w) + \mathrm{Diff}_y\,(u, v, w) + \mathrm{S}_y\,() - g\frac{\partial \zeta}{\partial y}. \tag{3}$$