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The approximate component mode synthesis special finite element method in two dimensions: Parallel implementation and numerical results

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ABSTRACT

A special finite element method based on approximate component mode synthesis (ACMS) was introduced in Hetmaniuk and Lehoucq (2010). ACMS was developed for second order elliptic partial differential equations with rough or highly varying coefficients. Here, a parallel implementation of ACMS is presented and parallel scalability issues are discussed for representative examples. Additionally, a parallel domain decomposition preconditioner (FETI-DP) is applied to solve the ACMS finite element system. Weak parallel scalability results for ACMS are presented for up to 1024 cores. Our numerical results also suggest a quadratic–logarithmic condition number bound for the preconditioned FETI-DP method applied to ACMS discretizations.

1. Introduction

We consider problems

$$\begin{aligned} -\nabla \cdot (A(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where the coefficient matrix A is *rough* or *highly varying* on a small scale. Such problems are also often referred to as multiscale problems. Multiscale problems are challenging to solve using standard finite element methods since very fine meshes are needed in order to resolve the features of the solution on the fine scale. The large number of degrees of freedom then leads to high demands with respect to memory and computational resources. One way to overcome these issues is to introduce methods which explicitly take into account the information on the small scale without resorting to a brute force discretization. By brute force discretization we refer to a very fine discretization using standard finite elements. This can be achieved, e.g., by including the coefficient information into the basis functions. Various approaches have been proposed in this field, including multiscale finite element (MsFEM) [1,2], mixed multiscale finite element [3], heterogeneous multiscale finite element [4], adaptive multiscale [5], generalized finite element [6–8], and component mode synthesis (CMS) [9,10] methods.

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The special finite element method considered in this paper was introduced by Hetmaniuk and Lehoucq in [11]. Additional theory has been provided in Hetmaniuk and Klawonn [12]. The method is designed as an approximation of the component mode synthesis method, using three different types of basis functions in order to do so. It combines *bubble-type* eigenmodes, *vertex-specific* energy minimizing extensions of nodal trace functions, and *coupling* edge-based eigenmodes. For a detailed description of the eigenmode problems, see Section 2.1, especially Formulae (2.13) and (2.14), and also Fig. 2. We will refer to these basis functions as ACMS[1] shape functions or coarse basis functions. An important additional property of these basis functions is, in contrast to the basis functions used in the CMS method, their local support. The resulting linear system is therefore sparse and the construction of the coarse basis functions parallelizes well. This will be pointed out again later.

Using the ACMS method, in order to achieve a comparable accuracy, the discretized system can be smaller by one to two orders of magnitude compared to a brute force discretization; see also Section 5.2.

The numerical construction of the ACMS shape functions can be computationally expensive but this step is easily parallelized. In this paper, we therefore investigate the computational cost of using an ACMS discretization in a parallel context. In our view, the parallel use of this method makes the most sense.

Although the linear systems arising from ACMS discretizations are much smaller compared to a brute force discretization, these systems can still be large and ill conditioned. We therefore combine the ACMS discretization with a parallel FETI-DP domain decomposition method as iterative solution method. We will show that the FETI-DP method applied to ACMS discretizations is numerically scalable and converges in a small number of iterations.

The remainder of the article is organized as follows. In Section 2, we first describe the Approximate Component Mode Synthesis (ACMS) method as it was introduced in Hetmaniuk and Lehoucq [11]. Then, we provide an error estimate for the ACMS method which was derived in Hetmaniuk and Klawonn [12]. Next, we describe our new parallel implementation of the ACMS method and conclude this section with an algorithmic description of the solution of the eigenvalue problems needed in the ACMS approach. In Section 3, we first provide a general algorithmic description of the FETI-DP domain decomposition method, first for standard finite elements and then applied to the ACMS special finite element discretization. In Section 4, we introduce several different model problems: the Poisson equation and two second order diffusion problems, one with a varying coefficient matrix and one with a highly oscillating coefficient matrix. We also consider two further model problems, whereas one is more heterogeneous than the ones considered before and the other is discontinuous with high coefficient jumps. Finally, in Section 5, we present numerical results using our parallel implementation of the ACMS method which show that the assembly of this special finite element method is parallel scalable. We also provide numerical results which show that the FETI-DP domain decomposition method applied to the ACMS discretization is weakly parallel scalable. Finally, in Section 6, a conclusion is provided.

2. Approximate component mode synthesis discretizations

2.1. Description of the method

The model problem (1.1) can be transformed into the variational formulation: find $u \in H_0^1(\Omega)$, such that

$$a(u, v) = L(v) \quad \forall v \in H_0^1(\Omega) \tag{2.1}$$

with the bilinear form and the linear functional

$$a(u, v) = \int_\Omega (\nabla u(x))^T A(x) \nabla v(x) dx \quad \text{and} \quad L(v) = \int_\Omega f(x) v(x) dx,$$

where $f \in L^2(\Omega)$. We assume that the matrix A is uniformly symmetric positive definite, and that it satisfies

$$0 < \alpha_{\min} \xi^T \xi \leq \xi^T A(x) \xi \leq \alpha_{\max} \xi^T \xi \quad \forall x \in \bar{\Omega} \text{ and } \xi \in \mathbb{R}^2 \setminus \{0\}$$

with constants $\alpha_{\min}, \alpha_{\max}$ independent of x. For the convergence theory developed in [12] it is assumed that the coefficients a_{ij} of the matrix $A = (a_{ij})$ are in $\mathcal{C}^1(\overline{\Omega})$. The ACMS method might be applied with less regularity assumptions on A but no convergence estimate is known so far for this case.

Although the method can be applied in a more general setting, for the theory in [12] a polygonal domain is assumed.

In order to define the finite element space of our special finite element method we consider a family $(\tau_h)_h$ of conforming partitions of Ω into triangles or convex quadrilaterals. The elements of the partition are assumed to be open sets, and the intersection of the closure of two distinct elements T and T' is either empty, a vertex, or a complete edge with two vertices. This partition introduces

$$\Gamma = \left(\bigcup_{T \in \tau_h} \partial T \right) \setminus \partial \Omega \tag{2.2}$$

[1] The letter "A" in ACMS stands for "Approximate" and emphasizes the approximation of a CMS technique.