Journal of Computational and Applied Mathematics 289 (2015) 134–141

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A robust linearization scheme for finite volume based discretizations for simulation of two-phase flow in porous media

Florin Adrian Radu [a,*], Jan Martin Nordbotten [a], Iuliu Sorin Pop [a,b], Kundan Kumar [a,c]

[a] Department of Mathematics, University of Bergen, P. O. Box 7800, N-5020 Bergen, Norway

[b] Department of Mathematics and Computer Science, Eindhoven University of Technology, P. O. Box 513, 5600 MB Eindhoven, The Netherlands

[c] ICES, University of Texas at Austin, 201 East 24th St, Austin, TX, United States

ARTICLE INFO

Article history:
Received 31 August 2014
Received in revised form 22 January 2015

Keywords:
Two-phase flow
Linearization schemes
Finite volume
MPFA
Convergence analysis

ABSTRACT

In this work we consider a mathematical model for two-phase flow in porous media. The fluids are assumed immiscible and incompressible and the solid matrix non-deformable. The mathematical model for the two-phase flow is written in terms of the global pressure and a complementary pressure (obtained by using the Kirchhoff transformation) as primary unknowns. For the spatial discretization, finite volumes have been used (more precisely the multi-point flux approximation method) and in time the backward Euler method has been employed. We present here a new linearization scheme for the nonlinear system arising after the temporal and spatial discretization. We show that the scheme is linearly convergent. Numerical experiments are presented that sustain the theoretical results.

1. Introduction

Water and soil pollution, CO_2 storage, enhanced oil recovery and nuclear waste management are typical examples of two-phase porous media flows with obvious high societal relevance. A crucial role in understanding two-phase flow in porous media is played by numerical simulations, including mathematical modeling and numerical methods.

Mathematical models of two-phase flow in porous media consist of coupled, nonlinear and possibly degenerate partial differential equations. This makes the design and implementation of efficient numerical schemes for two-phase flow in porous media a challenging task. Locally conservative discretizations such as finite volume [1–5] and mixed finite element [6,7] methods are popular spatial discretization as they alleviate many stability issues. Furthermore, often long time-scales are of interest in applications, so fully implicit temporal discretizations are, in general, preferred.

At each time step, the spatial and temporal discretizations thus lead to a large system of nonlinear equations. This system is usually solved by either Picard's method [8] or Newton's method [9,8,10–13]. The former is linearly convergent while the latter is quadratically convergent. The quadratic convergence of Newton's method comes at the price of only local convergence in solution space, however it remains a very powerful tool when applied to systems arising from discretization of parabolic equations. This is because, in this case, the starting iteration is chosen as the solution at the last time step and the

* Corresponding author.
E-mail addresses: florin.radu@math.uib.no (F.A. Radu), jan.nordbotten@math.uib.no (J.M. Nordbotten), i.pop@tue.nl (I.S. Pop), kundan.kumar@math.uib.no (K. Kumar).

http://dx.doi.org/10.1016/j.cam.2015.02.051

initial error can be controlled. In this way, by adjusting the time step size one ensures that the starting iteration is in the convergence region of the method. In order to apply Newton's method to degenerate problems one needs a further regularization step [11–13], which may alter the quality of the solution (in particular the mass balance). Moreover, the time-step restriction depends on the mesh diameter and on the size of the regularization step, which may be relatively restrictive in practice (see [12,13] for *a priori* derived convergence conditions for the Newton method when applied to transport equations). Thus, we identify two main concerns of Newton's method: the need of regularization for degenerate problems, as well as the time-step constraint implicitly imposed by the convergence region.

A possible improvement to Newton's method for degenerate problems is the semismooth Newton method [14]. This method is of a higher algorithmic complexity and requires additional reformulation of the model by adding so called complementary conditions. The semismooth Newton method can be applied to two-phase flow or multicomponent transport with much better results compared to Newton's method (see [14–16]).

In this work we propose and analyze a new linearization scheme for finite volume discretization schemes for two-phase flow. The proposed scheme is a monotone fixed point iteration [17,18]. We show the linear convergence and robustness of the scheme, subject to a relatively mild restriction on the time step size but independent of grid size. Moreover, the scheme does not involve the calculation of derivatives, which is an advantage over both Picard and Newton methods. Both analysis and numerical experiments indicate that the new scheme is a valuable alternative to Picard or Newton-type methods for solving two-phase flow in porous media.

The paper is structured as follows. In Section 2, we present the two-phase model considered here, describe the linearization scheme and show its convergence. The numerical results are given in Section 3, which show the applicability of the method. The paper ends with some concluding remarks in Section 4.

2. Mathematical model and discretization

We consider a simplified mathematical model for two-phase flow in porous media. The fluids are immiscible and incompressible and the solid matrix is non-deformable. The formulation adopted here uses the global pressure and a complementary pressure (obtained by using the Kirchhoff transformation) as primary unknowns (see [19–21]). For simplicity of exposition, we assume spatially homogeneous relative permeability and capillary pressure functions, and no gravity.

Throughout this paper we use common notations from functional analysis. The domain $\Omega \subset \mathbb{R}^d$, d being the dimension of the space, is open, bounded and with a Lipschitz continuous boundary. By C we mean a positive constant, not depending on the unknowns or the discretization parameters.

2.1. Governing equations

Mass (volume) balance:

$$\frac{\partial s}{\partial t} + \nabla \cdot \vec{q}_w = f_1(s). \tag{1}$$

Fractional phase flux

$$\vec{q}_w = -k\nabla\theta + f_w(s)\vec{q}. \tag{2}$$

Conservation of total phase volumes

$$\nabla \cdot \vec{q} = f_2(s). \tag{3}$$

Darcy's law for total flow

$$\vec{q} = -\lambda(s)k\nabla p. \tag{4}$$

The equations hold true in $\Omega \times [0, T]$, with T denoting the final time. The system is closed by an invertible relationship between the saturation s and the complementary pressure θ, i.e. $s = s(\theta)$ and constitutive laws for the various functions f_α, $\alpha \in \{1, 2, w\}$. The model is a reformulation of the two-phase system

$$\frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha \vec{q}_\alpha) = \Psi_\alpha, \quad \alpha = w, n, \tag{5}$$

$$\vec{q}_\alpha = -\frac{k_{r,\alpha}}{\mu_\alpha} k\nabla p_\alpha, \quad \alpha = w, n, \tag{6}$$

$$s_w + s_n = 1, \tag{7}$$

$$p_n - p_w = p^{cap}(s_w). \tag{8}$$

We denoted by w and n the wetting and the non-wetting phase, respectively. Here: the porosity ϕ, the densities ρ_w, ρ_n and the viscosities μ_w, μ_n are constants. Furthermore, there are no exchange terms between the phases, and the capillary pressure p^{cap} and the relative permeabilities $k_{r,w}$, $k_{r,n}$ are assumed known, monotone and Lipschitz continuous functions of the wetting phase saturation s_w. To transform the system (5)–(8) into the system (1)–(4) one introduces the global pressure