Journal of Computational and Applied Mathematics 289 (2015) 331–345

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Reconstruction of an unknown source parameter in a semilinear parabolic problem

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ARTICLE INFO

Article history:
Received 30 July 2014
Received in revised form 5 November 2014

MSC:
47J35
65M12
65M32

Keywords:
Semilinear parabolic equation
Inverse source problem

ABSTRACT

In this paper, a semilinear parabolic problem with an unknown time-dependent source function $p(t)$ is studied. This missing parameter is reconstructed from a given measurement of the total energy/mass in the domain. The existence and uniqueness of a solution in suitable function spaces is established under minimal regularity assumptions on the data. A numerical time-discrete scheme to approximate the unique weak solution and the unknown source parameter is designed and convergence of the approximations is proved. Finally, the theoretically obtained results are supported by a numerical experiment.

1. Introduction

Direct and inverse problems of parabolic type containing a source control parameter and/or an integral over (a subset of) the spatial domain of a function of the unknown solution are often used to model physical phenomena. The integral may appear in the boundary conditions and/or the governing partial differential equation itself. These problems arise in various fields of science and engineering, e.g. in thermoelasticity and in fluid flow [1], in heat transfer processes [2], in control theory [3,4], in chemical diffusion [5] and in vibration problems [6,7]. In the last decade, considerable efforts have been made in proving the existence and uniqueness of a solution to such a problems and in computing this solution numerically, cf. [8] and the references therein.

In this paper, we consider a sufficiently smooth domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, with boundary Γ. We study the inverse problem of finding a source parameter $p(t)$ and a function $u(t, \boldsymbol{x})$ obeying the following semilinear parabolic equation

$$\partial_t u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = p(t)f(\boldsymbol{x}) + g(u(t,\boldsymbol{x})) + r(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in (0,T] \times \Omega, \tag{1}$$

with initial condition

$$u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{2}$$

and Dirichlet boundary condition

$$u(t,\boldsymbol{x}) = \alpha(t,\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma, \tag{3}$$

that has a sufficiently continuous extension into $\overline{\Omega}$. The data functions f, g, r, u_0 and α are supposed to be known.

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http://dx.doi.org/10.1016/j.cam.2014.12.027
0377-0427/

To compensate the missing information about $p(t)$, which will be sought in $L_2(0,T)$, the total energy/mass in the domain is given by the integral measurement

$$\int_\Omega u(t,\boldsymbol{x}) = m(t), \quad t \in [0,T]. \tag{4}$$

An unknown time-dependent source function $p(t)$ also appears in paper [9], that deals with a one-dimensional linear heat equation along with mixed boundary conditions. In that article, $p(t)$ is recovered from the boundary measurement $u(0,t),\ t \in [0,T]$. An explicit formula for the Fréchet gradient of the cost functional at the measurement point is derived via the unique solution of the appropriate adjoint problem. The Conjugate Gradient Algorithm is then proposed for numerical solution of the inverse source problem. A similar problem to [9] has been studied by means of semigroups in [10,11]. For other references we refer the reader to [12–18].

A possible application of problem (1)–(4) is the identification of a function which expresses the warmth radiated by a source of which the position in space is given by the function $f(\boldsymbol{x})$. This control function $p(t)$ is determined such that the temperature distribution $u(t,\boldsymbol{x})$ fulfils the heat transfer process expressed by Eq. (1) under the conditions that the initial temperature is given by $u_0(\boldsymbol{x})$, that $\alpha(t,\boldsymbol{x})$ presents the temperature on the boundary of the domain and that the thermal energy in the whole space equals $m(t)$.

We now consider the continuous extension of α over the whole domain $\overline{\Omega}$ and we put $u = v + \alpha$. Then, (1)–(4) becomes

$$\begin{cases} \partial_t v(t,\boldsymbol{x}) - \Delta v(t,\boldsymbol{x}) = p(t)f(\boldsymbol{x}) + g(v+\alpha) + r(t,\boldsymbol{x}) - \partial_t\alpha(t,\boldsymbol{x}) + \Delta\alpha(t,\boldsymbol{x}), & (t,\boldsymbol{x}) \in (0,T] \times \Omega, \\ v(0,\boldsymbol{x}) = u_0(\boldsymbol{x}) - \alpha(0,\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ v(t,\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Gamma, \\ \displaystyle\int_\Omega v(t,\boldsymbol{x}) = m(t) - \int_\Omega \alpha(t,\boldsymbol{x}), & t \in [0,T]. \end{cases} \tag{5}$$

If (5) has a unique weak solution v in a suitable function space, then $v+\alpha$ is a unique weak solution of (1)-(4). Therefore, it is sufficient to study a problem of the form

$$\begin{cases} \partial_t u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = p(t)f(\boldsymbol{x}) + g(u) + r(t,\boldsymbol{x}), & (t,\boldsymbol{x}) \in (0,T] \times \Omega, \\ u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u(t,\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Gamma, \\ \displaystyle\int_\Omega u(t,\boldsymbol{x}) = m(t), & t \in [0,T]. \end{cases} \tag{6}$$

Throughout the rest of this paper we assume that $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, $f \in L_2(\Omega)$, $r : [0,T] \to L_2(\Omega)$ and $m : [0,T] \to \mathbb{R}$. Moreover, we will write $\boldsymbol{n}$ for the outward unit vector on Γ.

The well-posedness of problem (6) for a linear setting ($g \equiv 0$) has been addressed in [8]. The results for nonlinear cases are known for small T, i.e. locally in time—see also [8]. The added value of this paper relies on the global (in time) solvability of the problem and on the designed numerical scheme for approximations. The organization is as follows: in Section 2, we first eliminate the unknown source function p involving the integral condition (4). Next, we present a variational formulation of the resulting problem with u as the only unknown and we prove uniqueness of a weak solution to this problem. In Section 3, a numerical time-discrete approximation scheme is described and necessary a priori estimates are derived. Section 4 is devoted to the construction of a sequence of approximations for both the unknown function u and the source parameter p. In the same section, we prove that the limits (as the time step tends to zero) of these sequences in suitable function spaces fulfil the variational formulation. Finally, the results of a numerical experiment are presented in the last section to support the theoretically obtained convergence.

Remark. The values C, ε and C_ε are considered to be generic and positive constants (independent of the discretization parameter), where ε is arbitrarily small and C_ε arbitrarily large, i.e. $C_\varepsilon = C\left(\frac{1}{\varepsilon}\right)$. We will use the same notation for different constants, but the meaning will be clear from the context.

2. Variational formulation and uniqueness

For ease of exposition we will study problem (6). We denote by (u,v) the usual L_2-inner product of real-valued functions u and v in Ω, i.e. $(u,v) = \int_\Omega u \cdot v$ and $\|v\| = \sqrt{(v,v)}$. The L_2-inner product on the boundary Γ will be written as $(u,v)_\Gamma = \int_\Gamma u \cdot v$. We introduce the space of test functions

$$V := \left\{\varphi \in H^2(\Omega)\ ;\ \varphi_{|\Gamma} = 0\right\} = H^2(\Omega) \cap H_0^1(\Omega). \tag{7}$$

This space is equipped with the norm induced by the natural norm $\|\varphi\|_{2,2} := \sqrt{\sum_{|\alpha|\leqslant 2} \|D^\alpha\varphi\|^2}$ from $H^2(\Omega)$. The equivalence of this norm to the norm $\|\Delta\varphi\| =: \|\varphi\|_V$ in V is proven in the following theorem, which is based on classical regularity results for elliptic PDEs.

Theorem 1 (*Equivalence of Norms in V*). *$\|\cdot\|_{2,2}$ and $\|\cdot\|_V$ are equivalent norms in V.*