Journal of Computational and Applied Mathematics 289 (2015) 400–411

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

One-block method for computing the generalized stress intensity factors for Laplace's equation on a square with a slit and on an L-shaped domain

A.A. Dosiyev*, S.C. Buranay

Department of Mathematics, Eastern Mediterranean University, Gazimagusa, KKTC, Mersin 10, Turkey

ARTICLE INFO

Article history:
Received 23 July 2014
Received in revised form 6 November 2014

Keywords:
Laplace equation
Corner singularity
Block method
Stress intensity factor

ABSTRACT

A highly accurate approximation for the coefficients of the series expansion of the solution of Laplace's equation around the singular vertex, which are called the generalized stress intensity factors (GSIFs), is obtained by One-Block Method. The method is demonstrated for the slit problem, which has a strong singularity, and for the popular problem on an L-shaped domain. Furthermore, the extremely accurate series segment solution for each of the problems is obtained by taking an appropriate number of calculated GSIFs. Numerical results are presented through tables and figures.

1. Introduction

In many applied problems governed by elliptic type partial differential equations, boundary singularities arise due to nonsmooth geometric boundaries (reentrant corner), or by abrupt changes in the type of boundary conditions. It is already known that, without taking into consideration these singularities, the standard approximate methods can become very inaccurate, or at least, very costly. An exhaustive survey about the efficient approaches is provided in [1–3] and the references therein.

The behavior of the solution u of the two dimensional Laplace equation in the vicinity of a singular point, when the boundaries sharing this point are not curved, is given by

$$u(r,\theta) = \sum_{j=0}^{\infty} a_j r^{\mu_j} f_j(\theta), \tag{1}$$

where (r,θ) is the polar coordinate centered at the singular point, and $\mu_j, f_j(\theta)$ are determined by geometry and the boundary conditions along the boundaries sharing the singular point. The unknown constants a_j are often called generalized stress intensity factors (GSIFs) in theory of elasticity or generalized flux intensity factors (GFIFs) in heat transfer problems [4]. These coefficients have great importance in many applications, especially in fracture mechanics. The methods for the approximation of the coefficients a_j can be divided into two groups: (i) the post-processing approach in the finite element or finite difference methods (see [5–10]), in the block method (see [11–13]) and in the block-grid method [14–16] (ii) directly calculated methods, without finding approximate solution of the boundary value problem (see [4,1,17–20] and the references therein).

It is obvious that if the goal of computation is the calculation of the GSIFs/GFIFs, then the directly calculated methods become more preferable. In the last decade Trefftz method, which is named in [21] as the Boundary Approximation Method

* Corresponding author.
E-mail addresses: adiguzel.dosiyev@emu.edu.tr (A.A. Dosiyev), suzan.buranay@emu.edu.tr (S.C. Buranay).

http://dx.doi.org/10.1016/j.cam.2014.11.029
0377-0427/

(BAM) has recently been intensively studied. In BAMs, the solution is approximated over the entire problem domain as a linear combination of certain particular solutions of the governing equation. Thus, the boundary condition enforcement is necessary for the derivation of the unknown coefficients in the linear combination.

Special BAMs are developed for problems with a boundary singularity (see [20]). If the local asymptotic expansion (1) in the vicinity of a singular point is known and converges over the entire solution domain, then in Special BAMs the following approximation of the solution is taken

$$u^N(r,\theta)=\sum_{j=0}^{N} a_j^{\varepsilon} r^{\mu_j} f_j(\theta), \tag{2}$$

where a_j^{ε} are the approximations of the singular coefficients a_j, and for the enforcement of the essential boundary conditions different approaches in the various special BAMs are used. [22,4] employed least-squares techniques, and [23,24] employed Lagrange multipliers. For other approaches see [21].

The Special BAMs have several advantages: (a) the dimension of the problem is reduced by one; (b) GFIFs are computed explicitly; (c) the computed GFIFs converge to the exact values exponentially as N increases. Nevertheless, the numbers of convergent significant digits decrease and consequently the accuracy of the approximate solution does not increase as N increases. This happens because the large value of N causes serious difficulties due to the ill conditioning of the associated least-squares matrices (see [20]).

Finally, we mention the papers [11–13], in which by using the one block version of the block method (BM) given in [25,26] the highly accurate results are obtained for the solution of the Motz problem, cracked-beam problem and problems in L shaped domains respectively.

In this paper, we have developed the one block version of the block method to compute GFIFs with high accuracy not only for the first few coefficients, but for a sufficient number of them to get a highly accurate approximate solution as a series segment (2) (in particular for all of the coefficients used in BAMs [17,20]).

The paper is organized as follows: In Section 2 the implementation of the one-block method is described. In Section 3, we present the integral representations of the exact solution of the boundary value problem for Laplace's equation on a sector with central angle $\alpha\pi$, $1/2<\alpha\leq 2$ and with radius r_0, and the exact values of the coefficients a_j in (1) through the solution value on the curve $\{(r,\theta): r=r_0,\ 0\leq\theta\leq\alpha\pi\}$. The solutions of the slit problem (Section 4) and of the problem in an L-shaped domain (Section 5) are extended to the sector. For each problem a highly accurate formula for the GSIFs, by using the one block method, is given and the numerical results are presented. The approximate solution of the problems are presented using the finite sum of the series representations through the calculated coefficients depending on the number N of the GSIFs, and maximum errors are provided. Moreover, for the slit problem which has a stronger singularity, the shapes of the solution u and its derivatives u_x, u_{xx}, and u_{xy}, by finite sum and by the BM are drawn to display their singular behavior.

2. The block and one-block methods

The *block method* was proposed and justified in [25,26] by E.A. Volkov. This method for the boundary value problem of Laplace's equation on polygons is constructed through the following steps:

(i) The polygonal domain is covered by a finite number of overlapping subdomains, which are sectors, half-discs and discs, called extended blocks. Then, for each extended block, the basic block with a radius smaller than the radius of the corresponding extended block is chosen. Moreover, the union of the closed basic blocks should give the closed polygonal domain.

(ii) The approximate solution of the boundary value problem on each of the basic block is defined as a composite mid-point quadrature approximation of the integral representation of a harmonic function of the type of Poisson's integral. This approximate solution converges exponentially with respect to the number of quadrature nodes.

In the *one-block method* the solution of the boundary value problem is harmonically extended to a sector with the center at the singular vertex of the polygon. This sector is taken as an extended block, whereas the given polygon is used as a basic block (see Figs. 1 and 6). Taking the extension formulae (see (18) in Section 4 for the slit problem, and (29) in Section 5 for the L-shaped problem) into account, in order to obtain the solution values at the quadrature nodes for the approximation of the integral representation, the well-conditioned system of algebraic equations is obtained. In one-block method, the solution of this system is used to give an explicit formula for the approximate solution as a quadrature approximation of the integral representation.

In the following sections of this paper the method for the highly accurate approximation of the GSIFs which is based on the one-block version of the block method is proposed.

3. Boundary value problem on a sectorial domain

Let

$$T=T(r_0)=\{(r,\theta): 0<r<r_0,\quad 0<\theta<\alpha\pi, 0<\alpha\leq 2\}, \tag{3}$$

be a sector, and let $\Gamma^*=\{(r,\theta): r=r_0, 0\leq\theta\leq\alpha\pi\}$, $\Gamma_0=\{(r,\theta): 0<r<r_0, \theta=\alpha\pi\}$, $\Gamma_1=\{(r,\theta): 0<r<r_0, \theta=0\}$.