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Determination of a solely time-dependent source in a semilinear parabolic problem by means of boundary measurements☆

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ABSTRACT

A semilinear parabolic problem of second order with an unknown solely time-dependent source term is studied. The missing source is recovered from an additional integral measurement over the boundary. The global in time existence, uniqueness as well as the regularity of a solution are addressed. A new numerical scheme based on Rothe's method is designed and convergence of iterates towards the exact solution is shown.

1. Introduction

Inverse problems (IPs) usually lead to mathematical models that are ill-posed in the sense of Hadamard. Many of them do not have a solution in the strict classical sense, or if there is a solution, it might not be unique or might not depend continuously on the data. To prove global (in time) existence and uniqueness of a solution turn out to be a serious task. Another important goal in IPs is their solvability and description of a constructive algorithm for finding a solution. The standard algorithms for IPs start with suitable parametrization and they involve continuous dependence of a parametrized solution on the parameter. A cost functional capturing the error between parametrized and exact solutions at a given measurement place is minimized in appropriate function spaces. The common disadvantage of this approach is lack of convexity of the cost functional, which can be remediated by an appropriate regularization – cf. e.g. [1–3] – based on adding a suitable term to the functional in order to guarantee its convexity, ensuring the existence of a unique solution to the minimization problem by means of the theory of monotone operators [4,5]. This later problem can be solved numerically by adequate approximation techniques, such as the steepest descend, Ritz or Newton or Levenberg–Marquardt method, see e.g. [6,7].

We are interested in determining the unknown couple (u, h) obeying the following semilinear parabolic problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = h(t)f(x) + \alpha(u(x,t)) + \beta(x,t) & \text{in } \Omega \times (0,T), \\ \nabla u(x,t) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^d,\ d \geq 1$ is a bounded domain with a sufficiently smooth boundary Γ. The symbol $\boldsymbol{\nu}$ denotes the outer normal vector associated with Γ. The data functions u_0, f, α, β are given and $T > 0$. The unknown purely time-dependent source term $h(t)$ will be recovered from the following (non-invasive) measurement on the boundary

$$m(t) = \int_\Gamma u(x, t)\mathrm{d}x, \quad t \in [0, T]. \tag{2}$$

Recovery of an unknown source belongs to hot topics in inverse problems. If the source solely depends on the space variable, one needs an additional space measurement (e.g. solution at the final time), cf. [8–17]. For purely time-dependent source a supplementary time-dependent measurement is needed, cf. [18–20]. This means that both kinds of inverse source problems (ISPs) need totally different additional data. The integral overdetermination is frequently used in various IPs for evolutionary problems, cf. [8,21,22] and the references therein. The integral is usually taken over the whole domain (or over a sub-domain). To get such a measurement is not always obvious. We consider the integration just over the boundary Γ in (2).

The goal of this paper is to address the well-posedness of the ISP, to study the regularity of a solution and to describe a constructive way for finding it. The added value of this paper relies on reformulating the ISP into an appropriate direct formulation. This means that we are looking on the ISP as on a system with two unknowns (u, h). We eliminate h from (1) by (2). The proposed numerical scheme involves the semi-discretization in time by Rothe's method cf. [23]. We prove the existence of approximations at each time step of the time partitioning and we establish some stability results. The convergence of iterates towards the exact solution is obtained by arguments of functional analysis. Finally, we discuss uniqueness of the ISP.

Notations. Denote by $(\cdot, \cdot)$ the standard inner product of $L^2(\Omega)$ and $\|\cdot\|$ its induced norm. When working at the boundary Γ we use a similar notation, namely $(\cdot, \cdot)_\Gamma$, $L^2(\Gamma)$ and $\|\cdot\|_\Gamma$. By $C\,([0, T], X)$ we denote the set of abstract functions $w : [0, T] \to X$ equipped with the usual norm $\max_{t\in[0,T]} \|\cdot\|_X$ and $L^p\,((0, T), X)$ is furnished with the norm $\left(\int_0^T \|\cdot\|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}}$ with $p > 1$, cf. [24]. The symbol X^* stands for the dual space to X. As is usual in papers of this sort, C, ε and C_ε will denote generic positive constants depending only on a priori known quantities, where ε is small and $C_\varepsilon = C\left(\varepsilon^{-1}\right)$ is large.

Take any function $\varphi \in H^1(\Omega)$, and derive from (1) after integration over Ω and involving the Green theorem that

$$(\partial_t u, \varphi) + (\nabla u, \nabla \varphi) = h\,(f, \varphi) + (\alpha(u), \varphi) + (\beta, \varphi)\,. \tag{P}$$

Integrating (1) over Γ and taking into account the measurement (2) we have

$$m' - \int_\Gamma \Delta u = h \int_\Gamma f + \int_\Gamma \alpha(u) + \int_\Gamma \beta. \tag{MP}$$

The relations (P) and (MP) represent the variational formulation of (1) and (2).

2. Time discretization

In Rothe's method [23], a time-dependent problem is approximated by a sequence of elliptic tasks which have to be solved successively with increasing time step. Rothe's method can be also used for determination of the unknown time dependent source h. For ease of explanation we consider an equidistant time-partitioning of the time frame $[0, T]$ with a step $\tau = T/n$, for any $n \in \mathbb{N}$. We use the notation $t_i = i\tau$ and for any function z we write

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

Consider a system with unknowns (u_i, h_i) for $i = 1, \ldots, n$. At time t_i we infer from (P) by the backward Euler scheme

$$(\delta u_i, \varphi) + (\nabla u_i, \nabla \varphi) = h_i\,(f, \varphi) + (\alpha(u_{i-1}), \varphi) + (\beta_i, \varphi)\,. \tag{DPi}$$

Considering u_{i-1} in the right-hand side makes (DPi) linear in u_i. From (MP) we obtain

$$m_i' - \int_\Gamma \Delta u_{i-1} = h_i \int_\Gamma f + \int_\Gamma \alpha(u_{i-1}) + \int_\Gamma \beta_i. \tag{DMPi}$$

The decoupling of u_i and h_i has been achieved by considering u_{i-1} in (DMPi). Note that for a given $i \in \{1, \ldots, n\}$ we solve first equation (DMPi) and then (DPi). Further we increase i to $i + 1$.

Let us introduce the following space of test functions

$$\boldsymbol{V} = \{\varphi : \Omega \to \mathbb{R};\ \ \|\varphi\| + \|\nabla\varphi\| + \|\Delta\varphi\| + \|\nabla\Delta\varphi\| < \infty\}\,,$$

which is suitable for our purposes. We will seek u within this space. Let us note that we have to work in sufficiently regular function space in order to keep h depending on $\int_\Gamma \Delta u$ under control, cf. (MP).

Lemma 2.1. *Let $m' \in C([0, T])$, $\beta \in C\left([0, T], H^1(\Omega)\right)$, $f \in H^1(\Omega)$, $\int_\Gamma f \neq 0$, α is global Lipschitz continuous. Assume that $u_0 \in \boldsymbol{V}$. Then for each $i \in \{1, \ldots, n\}$ there exists a unique couple $(u_i, h_i) \in \boldsymbol{V} \times \mathbb{R}$ solving (DPi) and (DMPi).*