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Applying the combined integral method to two-phase Stefan problems with delayed onset of phase change

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ABSTRACT

In this paper the combined integral method (CIM) is applied to non-classical two-phase Stefan problems with delayed onset of phase change. This can occur if the phase change is caused by a heat-flux or Robin boundary condition. The method requires choosing an approximating function, typically a polynomial, but it is not clear what should be used as the exponent in the highest order term. Previous studies have determined exponents either from exact solutions or from expansions valid over short time scales; neither approach is satisfactory and can be very inaccurate for larger times. We combined the heat balance and refined integral methods to determine this exponent as part of the solution process, allowing the exponent to be time-dependent. From comparing the approximate solutions with numerical and exact analytical solutions whenever possible, we show that the CIM greatly improves the accuracy on standard heat balance integral methods, without detracting from its simplicity.

1. Introduction

Phase-change, or Stefan, problems in which a material melts or solidifies occur in a wide variety of natural and industrial processes. Mathematically, these are special cases of moving-boundary problems, where the location of the front between the liquid and solid is not known a priori, but must be determined as part of the solution. In the classical one-dimensional Stefan problem, which is most studied in the literature, the phase change occurs due to a constant heating or cooling temperature. However, it is more practically relevant to consider the situation where a heat flux or convective boundary condition is imposed.

In this paper we apply a combination of conventional heat balance and refined integral methods (HBIMs and RIMs, respectively) to the two-phase Stefan problem where the second phase first appears only after a finite delay time. The analysis is based on a solidification problem but it could easily be recast as a melting problem. Our aim is to develop an integral method that provides a more accurate description of the process. The main advantage of these integral methods is that they significantly reduce the complexity of the problem. In the first stage of the process, called the pre-solidification stage, there is only a liquid region and thus no moving boundary. For both a heat-flux and Robin boundary condition we can write down an exact solution; this can then be compared with the CIM solution. The method involves introducing a heat penetration depth $\delta(t)$, whereby for $x > \delta$ the temperature change above the initial temperature is assumed to be negligible. Then an approximating function is defined for the temperature, typically a polynomial, and by applying sufficient boundary conditions at $x = 0$ and $x = \delta$, all the unknown coefficients can be determined in terms of the unknown function δ. Finally, the governing heat equation is integrated for $x \in [0, \delta]$ to produce a heat balance integral, leading to an ordinary differential

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equation to solve for δ [1,2]. This stage ends when the temperature at $x = 0$ has reduced to the solidification temperature. The second stage of the process, which consists of both the liquid and solid phases, is governed by two heat equations coupled to a Stefan condition. In this stage we specify polynomial profiles in both the liquid and solid regions, and this allows us to determine the location of the solidification front as it moves with time.

Goodman [3] originally proposed the *heat balance integral method* (HBIM) for solving thermal and Stefan problems, which was an adaptation of the Karman–Pohlhausen integral method [4] for analysing boundary layers; see [5] for a translated account of this work. Since exact solutions have been found for many problems in heat transfer, the HBIM has made its greatest impact on Stefan problems, where very few exact solutions exist. Their popularity arises from their simplicity and the fact that they produce analytic solutions for a wide range of problems and parameter values. However, it should be noted that they are not always as accurate as numerical solutions. In the last few years Mitchell and Myers [6–9,2,10–14] have published a series of papers devoted to integral methods applied to a variety of thermal and moving boundary problems. A comprehensive review of the HBIM and its variants is given in [9], along with a discussion of its disadvantages. Mitchell and Myers have also investigated ways to improve the HBIM by using an unknown exponent in the approximating profile to describe the dependent variable (usually the temperature) [2,11–13], and these have always given more accurate solutions than the classic HBIM solutions. The CIM solution combines the HBIM and RIM solutions and allows the exponent of the approximating function to be time-dependent [2,13]. Myers [11,12] has employed an alternative method to determine the exponent by minimising an error function. Whilst this can sometimes improve on the CIM in certain cases, it does not easily allow for situations when the exponent is time-dependent and the minimisation can be complicated. In addition, it is not clear how to apply the Myers method in situations with two-stages, or even three-stages if the domain is finite. This is discussed in more detail in Section 4.

A recent paper by Mitchell [1] applies the CIM to the two-stage, one-phase, ablation problem, which is defined as the removal of a material from the surface of an object by vaporisation, chipping or other erosive processes. The CIM gives dramatically improved results to standard HBIM methods, by showing that the exponent in the approximating function was time-dependent, and that using a constant value was very inaccurate. This improvement motivated considering an application with two phases which also had a delay in the onset of phase change. This has only previously been studied numerically by Mitchell and Vynnycky [15], and this was in the restricted case where the domain of the liquid region was always assumed to be semi-infinite. Whilst we also treat this case in detail here, we also consider finite domains, which are far more realistic but more complicated to analyse.

The layout of this paper is as follows. In Section 2 we formulate the governing non-dimensional equations to describe the two-phase Stefan problem with the delayed onset of phase change, using a heat-flux boundary condition. In Section 3 we apply the CIM to the problem on a semi-infinite domain and discuss the Robin boundary condition, motivating how the boundary condition could arise from including how to model the surface that the liquid region rests upon. Section 4 extends the analysis to the case where the domain of the liquid is finite, and discusses the different cases that arise depending on the size of the domain. All the integral method solutions are compared with a numerical solution, the details of which are given in the Appendix. Finally, in Section 5 we draw conclusions.

2. Mathematical formulation

Consider the cooling of a liquid, occupying the half-plane $y > 0$, that is initially at a temperature, T_{hot}, which is greater than its melting temperature, T_{melt}, and which is cooled at $y = 0$ for time $t > 0$ by an applied heat flux $Q(t)$. After cooling commences, the temperature of the liquid decreases until some time t_m, at which stage solid begins to form at $y = 0$; subsequently, solid occupies the region $0 \le y \le y_m(t)$ and liquid occupies $y > y_m(t)$, where $y_m(t)$ denotes the location of the solidification front. A schematic is shown in Fig. 1.

Assuming the material properties of the solid and liquid phases to be constant, the governing equations are as follows. For $0 < t < t_m$ and $y > 0$, and then $t > t_m$ and $y > y_m(t)$ we have

$$\rho_l c_{pl} \frac{\partial T_l}{\partial t} = k_l \frac{\partial^2 T_l}{\partial y^2}, \tag{1}$$

where T_l is the liquid temperature, k_l is the thermal conductivity of the liquid, c_{pl} is its specific heat capacity and ρ_l its density. For $t > t_m$ and $0 < y < y_m(t)$, we have

$$\rho_s c_{ps} \frac{\partial T_s}{\partial t} = k_s \frac{\partial^2 T_s}{\partial y^2}, \tag{2}$$

where T_s is the solid temperature, k_s is the thermal conductivity of the solid, c_{ps} is its specific heat capacity and ρ_s its density. We assume henceforth that $\rho_l = \rho_s = \rho$.

For boundary conditions, we have, at $y = y_m(t)$,

$$T_s = T_l = T_{melt}, \tag{3}$$

and the Stefan condition,

$$k_s \frac{\partial T_s}{\partial y} - k_l \frac{\partial T_l}{\partial y} = \rho \Delta H_f \frac{dy_m}{dt}, \tag{4}$$