Journal of Computational and Applied Mathematics 281 (2015) 169–181

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Estimate of Morse index of cooperative elliptic systems and its application to spatial vector solitons

Xianjin Chen [a,*], Jianxin Zhou [b]

[a] School of Mathematical Science, University of Science and Technology of China, Hefei 230026, PR China
[b] Department of Mathematics, Texas A&M University, College Station, TX 77843, United States

ARTICLE INFO

Article history:
Received 26 June 2014

MSC:
35A40
35A15
58E05
58E30

Keywords:
Cooperative elliptic system
Multiple solutions
Local min-orthogonal method
Saddle point
Spatial vector soliton

ABSTRACT

Local instability index of unstable solutions to single partial differential equations (PDEs) by a local minimax method (LMMM) was established in Zhou (2005). It is known that the local min-orthogonal method (LMOM) which was first proposed in Zhou (2004) and then further developed in Chen et al. (2008) can find more general unstable solutions to both single PDEs and cooperative elliptic systems. This paper is to carry out instability analysis of unstable solutions by LMOM, to which an infinite-dimensional functional space can be decomposed as a direct sum of a finite-dimensional subspace and its orthogonal complement. A Morse index approach is developed to show that with LMOM, instability behavior of a solution in such infinite-dimensional complement subspace can be totally determined. Usual instability analysis in an entire space is then reduced to analysis in its finite-dimensional subspace, for which a corresponding matrix decomposition is proposed to analyze a solution's instability behavior. Estimates of Morse index are also established. Finally, numerical examples of both 2- and 3-component cooperative systems arising in nonlinear optics are carried out for spatial vector solitons, whose local instabilities are numerically confirmed by the new estimates. Certain important properties of the examples are also verified or presented.

1. Introduction

Multiple solution phenomena appear in many applications; e.g., electronically (or laser) induced, highly/multiply excited states (corresponding to saddle points of certain functionals mathematically) can now be observed or obtained [1–7] in many fields such as quantum mechanics, condensed matter physics, dynamics of biomolecules, and nonlinear optics. Those highly excited states are usually very unstable and tend to decay via different manners into lower-lying excited ones or the ground states under small perturbations. Meanwhile, all those excited states enjoy a wide variety of configurations and maneuverabilities, which may lead to various applications in future. With new atomic, optical or synchrotronic technologies, scientists now are able to successfully reach or trap them and search for their potential applications [8,2,9–14,7]. These new technologies and advanced studies are changing people's traditional views of unstable solutions and starting to draw more and more attention from both scientists and engineers. As a result, there is a growing need to develop more efficient and reliable numerical methods for finding those multiple excited states and for analyzing their instability as well. A local minimax method (LMMM) [15–17] and a local min-orthogonal method (LMOM) [18,19] were developed to numerically

* Corresponding author.
E-mail addresses: chenxjin@ustc.edu.cn (X. Chen), jzhou@math.tamu.edu (J. Zhou).

http://dx.doi.org/10.1016/j.cam.2014.11.046

find multiple unstable solutions for several nonlinear PDEs and PDE systems. Instability analysis of saddle points including estimates of their Morse indices was established in [20,21] for single PDEs under a local minimax characterization. On the other hand, analogous analysis has not been implemented for PDE systems under a local min-orthogonal characterization. The purpose of this paper is to carry out local instability analysis of multiple co-existing solutions to cooperative elliptic systems. Compared with their counterparts in the single equation case, systems usually carry a much richer complexity and a more intriguing dynamics and can be classified in many different ways, see also [18,22].

Consider the following cooperative system

$$\begin{cases} -\Delta u = F_u(x, u, v), & x \in \Omega, \\ -\Delta v = F_v(x, u, v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ ($N \geq 1$), $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is of class $\mathcal{C}^2$ in the variables (u, v) such that

(A1) $|\nabla F(x, z)| \leq C(1+|z|^{p-1})$, $\forall z \in \mathbb{R}^2$, a.e. $x \in \Omega$, for some constants $C > 0$ and $2 < p < \frac{2N}{N-2}$ if $N \geq 3$ or $2 < p < +\infty$ if $N = 1, 2$ (subcritical growth [23,24]),

(A2) $F_u(x, 0, v) \equiv F_v(x, u, 0) \equiv 0$ (homogeneity),

(A3) when $|(u, v)| \to 0$, $F(x, u, v) = \frac{\alpha}{2}u^2 + \frac{\beta}{2}v^2 + o(|(u, v)|^2)$ for some constants α, β s.t. $\max\{\alpha, \beta\} < \sigma_1$, the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Then, weak solutions to (1.1) correspond to critical points of the variational functional

$$J(u, v) = \int_\Omega \left\{ \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2) - F(x, u, v) \right\} dx. \tag{1.2}$$

It is easy to see that $(0, 0)$ is a local minimum of J under conditions (A2) and (A3). Recall that for a functional $J \in \mathcal{C}^1(H, \mathbb{R})$ in a Hilbert space H, a point $w^* \in H$ is called a *critical point* of J if $J'(w^*) = 0$, where J' denotes the first Frechet derivative of J. A critical point w^* of J is called *nondegenerate* if the second Frechet derivative $J''(w^*)$ is not singular; otherwise, w^* is called *degenerate*. The most studied critical points are the local extrema. Classical *calculus of variations* and traditional numerical (variational) methods focus on finding such stable solutions. Critical points that are not local extrema are called *saddle points*; they appear as unstable equilibria or transient excited states in physical systems.

Multicomponent solitons have recently witnessed [2–4,9–11] a renewed interest in the areas of condensed matter physics, dynamics of biomolecules, nonlinear optics, etc. For example, after a number of experimental observations of self-guided light beams in various types of nonlinear bulk media (e.g., photorefractive crystals, photonic crystal fibers) were reported, the study of optical spatial solitons and their interactions have become an active research area in nonlinear optics, see [2–4,25,5,10,11]. As mentioned in [2], "Optical spatial solitons are not only interesting in themselves, but also because they exhibit many fascinating features such as particle-like interactions during collisions.... The general ideas behind multicomponent vector solitons proved invaluable for later developments". In nonlinear optics, it has been shown that several light beams can be combined to produce multicomponent self-trapped states, also called spatial vector solitons. Physically, these spatial vector solitons are localized "particlelike" nonlinear objects, i.e., self-trapped, self-guided light beams. Mathematically, they are standing (stationary) solitary wave solutions of certain nonlinear Schrödinger systems [3,4,25,5], see system (4.1) in Section 4, for example; in particular, when the systems are variational, they correspond to co-existing saddle points of certain functionals.

Existence and stability are always among the main concerns in nonlinear PDEs or PDE systems. Recently, existence of multiple solutions (including vector solitons) to several classes of semilinear and asymptotically linear elliptic systems has been confirmed experimentally (see [3,4,25,5] and the references therein) and/or established mathematically [26–28,24,29,30]. On the stability of vector solitons, "so far, numerical methods have been proved to be the only available tool for analyzing the mutually trapped states in the nonlinear regime, especially solitons with no radial symmetry (e.g., dipole or multipole vector solitons)" [25]. It has been observed that the dipole-mode vector solitons are much more stable than any other mode vector soliton including the vortex-mode vector soliton. They are "stable enough for experimental observation, ..., extremely robust, have a typical lifetime of several hundred diffraction lengths and survive a wide range of perturbations" [5]. On the other hand, from a mathematical point of view, all those vector solitons are saddle points and thus unstable. One objective of this paper is to try to mathematically measure local instabilities of those unstable solutions.

For a critical point u^* of J in H, assume $J''(u^*) : H \to H$ is a self-adjoint Fredholm operator. According to the spectral theory, H has an orthogonal spectral decomposition

$$H = H^- \oplus H^0 \oplus H^+ \tag{1.3}$$

where H^-, H^0, H^+ are respectively the maximum negative definite, the null and the maximum positive definite subspaces of $J''(u^*)$ in H with $\dim(H^0) < \infty$, and are invariant under $J''(u^*)$. By the Morse theory, the *Morse index* (MI) of u^* is $MI(u^*) = \dim(H^-)$. Evidently, a nondegenerate critical point u^* with $MI(u^*) = 0$ is a local minimum of J and hence is a stable solution; while a critical point u^* of J with $MI(u^*) > 0$ is an unstable solution. When u^* is nondegenerate, i.e., $H^0 = \{0\}$, the value $MI(u^*)$ can be used [31] to measure local instabilities of u^*. In other words, it can be used as a local instability index, i.e., the dimension of a maximum subspace of vectors in H along which the functional J decreases