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A survey on fuzzy fractional variational problems

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ABSTRACT

This paper focuses on introducing and studying generalized fuzzy Euler–Lagrange equation and fuzzy isoperimetric problem. According to the concept of Caputo-type fuzzy fractional derivative in the sense of the generalized fuzzy differentiability, we extend and establish some definitions on fuzzy fractional calculus of variation and provide some necessary conditions to obtain the fuzzy fractional Euler–Lagrange equation for both constrained and unconstrained fuzzy fractional variational problems. The fuzzy isoperimetric problem is also investigated in relation to the generalized fuzzy Euler–Lagrange. Finally, two examples are given to describe the proposed approach.

1. Introduction

During the past decades, calculus of variations and fractional calculus have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, system identification, control theory, finance, and fractional dynamics. Since the fractional theory has played a very significant role in many branches of science, numerous researchers have investigated some aspects of this field and generalized the classical results of integer order. Particularly, non-conservative Lagrangian, Hamiltonian, and other concepts of classical mechanics using fractional derivatives was first developed by Riewe in [1,2]. Afterward, Agrawal [3] combined calculus of variations and the concept of fractional derivative to develop Euler–Lagrange equations for fractional variational problems. We cite also Chau's works [4–9], who investigated about real-life applications of various contemporary soft computing techniques in this field.

Although the notion of fuzzy set is widely spread to various problems such as linear programming, optimization, differential equations and even fractional differential equations, investigating fuzzy variational problem is seldom available in the papers [10–12]. B. Farhadinia was the first one who introduced and established the concept of fuzzy variational problem in his work [12]. He obtained some necessary optimality conditions for fuzzy variational problems based on the Seikkala derivative for fuzzy-valued functions. Farhadinia's work was continued by Omid S. Fard et al. [10,11] in the sense of the Hukuhara and the generalized differentiability.

The most famous fuzzy fractional derivatives are Riemann–Liouville and Caputo and the most frequently used is the Caputo-type. In this paper, we restrict our attention to the Caputo-type fuzzy fractional derivative which first introduced by Mazandarani and Kamyad [13]. Following the definition of the Caputo-type fuzzy fractional derivatives [13], we intend to study and discuss the fuzzy fractional calculus of variations. Specifically, we will focus on providing some proper conditions that allow us to derive the necessary optimal conditions for both fuzzy constrained and unconstrained variational problems containing the fuzzy Caputo-type fractional derivatives.

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Here is a brief survey of the main contents. Some notations on the fuzzy numbers space, differentiability and integrability of a fuzzy mapping and the concept of the Caputo-type fuzzy fractional derivatives are stated in Section 2. In Sections 3 and 4, we establish the main results concerning the fuzzy Euler–Lagrange equations for the fuzzy unconstrained fractional variational problems and the modified fuzzy Euler–Lagrange conditions for the fuzzy constrained fractional variational problems, called the isoperimetric problem, in detail. In Section 5, two examples are given to illustrate the results. Finally, the paper will be concluded in Section 6.

2. Preliminaries

This section presents some definitions and basic concepts which will be used in this paper.

Definition 2.1. By $\mathbb{R}$, we denote the set of all real numbers and by E, the space of fuzzy numbers $\tilde{u}(x) : \mathbb{R} \to [0, 1]$, satisfying the following requirements:

1. $\tilde{u}(x)$ is normal, i.e. $\exists x_0 \in \mathbb{R}$ for which $\tilde{u}(x_0) = 1$,
2. $\tilde{u}(x)$ is fuzzy convex, i.e., $\forall x_1, x_2 \in \mathbb{R},\ \lambda \in [0, 1],\ \tilde{u}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\tilde{u}(x_1), \tilde{u}(x_2)\}$,
3. Supp $\tilde{u}(x) = \{x \in \mathbb{R} \mid \tilde{u}(x) \geq 0\}$ is the support of the $\tilde{u}(x)$ and its closure $cl(\text{Supp}\,\tilde{u}(x))$ is compact,
4. $\tilde{u}(x)$ is upper semi-continuous.

The α-cut set of a fuzzy number $\tilde{u}(x) \in E$ denoted by $[\tilde{u}(x)]^\alpha$, is defined as

$$[\tilde{u}(x)]^\alpha = \begin{cases} \{x \in \mathbb{R} \mid \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1; \\ cl(\text{Supp}\,\tilde{u}(x)), & \alpha = 0. \end{cases}$$

It is well known that the α-cut set of a fuzzy number is a closed and bounded interval $[\underline{u}^\alpha(x), \overline{u}^\alpha(x)]$, where $\underline{u}^\alpha(x)$ and $\overline{u}^\alpha(x)$ denote the left-hand endpoint of $[\tilde{u}(x)]^\alpha$ and the right-hand endpoint of $[\tilde{u}(x)]^\alpha$, respectively.

Definition 2.2. The parametric form of a fuzzy number $\tilde{u}(x)$ is a pair $\left(\underline{u}^\alpha(x), \overline{u}^\alpha(x)\right)$, of functions $\underline{u}^\alpha(x), \overline{u}^\alpha(x),\ 0 \leq \alpha \leq 1$, which satisfy the following conditions:

1. $\underline{u}^\alpha(x)$ is a monotonically increasing left continuous function.
2. $\overline{u}^\alpha(x)$ is a monotonically decreasing left continuous function.
3. $\underline{u}^\alpha(x) \leq \overline{u}^\alpha(x),\ 0 \leq \alpha \leq 1$.

Addition $\tilde{u} + \tilde{v}$ and subtraction $\tilde{u} - \tilde{v}$ and scalar multiplication by k are defined as

$$[\tilde{u} + \tilde{v}]^\alpha = [\underline{u}^\alpha + \underline{v}^\alpha, \overline{u}^\alpha + \overline{v}^\alpha],$$
$$[\tilde{u} - \tilde{v}]^\alpha = [\underline{u}^\alpha - \overline{v}^\alpha, \overline{u}^\alpha - \underline{v}^\alpha],$$

and

$$\left[k\tilde{u}\right]^\alpha = \begin{cases} \left[k\underline{u}^\alpha, k\overline{u}^\alpha\right], & k \geq 0; \\ \left[k\overline{u}^\alpha, k\underline{u}^\alpha\right], & k < 0. \end{cases}$$

Remark 2.3. A function $f : A \to E,\ A \subseteq \mathbb{R}$ so called fuzzy-valued function. However, an arbitrary function f, where $f : A \to \mathbb{R},\ A \subseteq \mathbb{R}$ so called real-valued function (or crisp function). The α-cut representation of fuzzy-valued function f can be expressed by $[\tilde{f}(x)]^\alpha = [\underline{f}^\alpha(x), \overline{f}^\alpha(x)]$, $x \in A \subseteq \mathbb{R}$ and $0 \leq \alpha \leq 1$.

Definition 2.4. The Hausdorff distance between fuzzy numbers given by $d : E \times E \to [0, +\infty]$,

$$d(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\},$$

where $\tilde{u} = (\underline{u}^\alpha, \overline{u}^\alpha), \tilde{v} = (\underline{v}^\alpha, \overline{v}^\alpha) \subset \mathbb{R}$ is utilized in [14]. Then, it is easy to see that d is a metric in E and has the following properties (see [15]):

(1) $d(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) = d(\tilde{u}, \tilde{v}),\ \forall \tilde{u}, \tilde{v}, \tilde{w} \in E$,
(2) $d(k\tilde{u}, k\tilde{v}) = |k| d(\tilde{u}, \tilde{v}),\ \forall k \in \mathbb{R},\ \tilde{u}, \tilde{v} \in E$,
(3) $d(\tilde{u} + \tilde{v}, \tilde{w} + \tilde{e}) \leq d(\tilde{u}, \tilde{w}) + d(\tilde{v}, \tilde{e}),\ \forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in E$,
(4) (d, E) is a complete metric space.

Definition 2.5 (*[12] Partial Ordering*)**.** Let $\tilde{a}, \tilde{b} \in E$. We write $\tilde{a} \preceq \tilde{b}$, if $\underline{a}^\alpha \leq \underline{b}^\alpha$ and $\overline{a}^\alpha \leq \overline{b}^\alpha$ for all $\alpha \in [0, 1]$. We also write $\tilde{a} \prec \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and there exists an $\hat{\alpha} \in [0, 1]$ so that $\underline{a}^{\hat{\alpha}} < \underline{b}^{\hat{\alpha}}$ or $\overline{a}^{\hat{\alpha}} < \overline{b}^{\hat{\alpha}}$. Moreover, $\tilde{a} \approx \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$ or in the other words, $\tilde{a} \approx \tilde{b}$, if $\tilde{a}^\alpha = \tilde{b}^\alpha$ for all $\alpha \in [0, 1]$.

In the sequel, we say that $\tilde{a}, \tilde{b} \in E$ are comparable if either $\tilde{a} \preceq \tilde{b}$ or $\tilde{a} \succeq \tilde{b}$, and non-comparable otherwise.

Definition 2.6. Let $\tilde{u},\ \tilde{v} \in E$. If there exists $\tilde{w} \in E$ such that $\tilde{u} = \tilde{v} + \tilde{w}$, then $\tilde{w}$ is called the Hukuhara difference (H-difference) of $\tilde{u}$ and $\tilde{v}$, and it is denoted by $\tilde{u} \ominus \tilde{v}$. Note that $\tilde{u} \ominus \tilde{v} \neq \tilde{u} + (-1)\tilde{v}$.