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Numerical analysis of an energy-like minimization method to solve a parabolic Cauchy problem with noisy data

R. Rischette[a], T.N. Baranger[a,*], N. Debit[b]

[a] Université de Lyon, CNRS, LaMCoS UMR5259, Université Lyon 1, INSA-Lyon, 69621 Villeurbanne, France
[b] Université de Lyon, CNRS, Université Lyon 1, Institut Camille Jordan, F-69622 Villeurbanne Cedex, France

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ABSTRACT

This paper is concerned with solving the Cauchy problem for the parabolic equation by minimizing an energy-like error functional and by taking into account noisy Cauchy data. After giving some fundamental results, numerical convergence analysis of the energy-like minimization method is carried out and leads to an adapted stopping criteria depending on noise rate for the minimization process. Numerical experiments are performed and confirm the theoretical convergence order and the good behavior of the minimization process.

1. Introduction

The Cauchy problem considered here consists of solving a parabolic partial differential equation on a domain for which over-specified boundary conditions are given on a part of its boundary. It entails solving a data completion problem and identifying the missing boundary conditions on the remaining part of the boundary. This kind of problem is encountered in many industrial, engineering and biomedical applications.

Since J. Hadamard's works [1], the Cauchy problem is known to be ill-posed and considerable numerical instability may occur during the resolution process. It provides researchers with an interesting challenge for carrying out numerical procedures to approximate the solution of the Cauchy problem in the specific case of noisy data. Many theoretical and applied works have been dedicated to this subject, using iterative methods [2,3], regularization methods [4–6], quasi-reversibility methods [7] and minimal error methods [8–10].

In this paper, we focus on a method introduced in [11–16] based on the minimization of an energy-like functional. More precisely, we introduce two distinct fields, each of which provides one of the over specified data. They are therefore solutions of two well-posed problems. Next, an energy-like error functional is introduced to measure the gap between these two fields. If this gap exists and is unique, the Cauchy problem solution is obtained when the functional reaches its minimum. Then, the resolution of the ill-posed Cauchy problem is achieved by successive resolutions of well-posed problems. This method provides promising results. Nevertheless, like many other methods, it becomes unstable in the case of noisy data. To overcome this numerical instability, we propose an adequate stopping criterion parametrized by the noise rate deduced by numerical convergence analysis. This analysis has already been performed for elliptic Cauchy problems in [17].

* Corresponding author. Tel.: +33 72448131.
E-mail address: thouraya.baranger@univ-lyon1.fr (T.N. Baranger).

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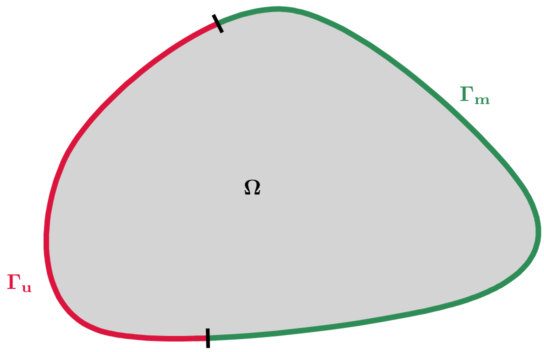

Fig. 1. An example of geometry.

The outline of the paper is as follows. In Section 2, we give the Cauchy problem and report classical theoretical results. In Section 3, we formulate the Cauchy problem as a data completion problem and introduce the related minimization problem. In Sections 4 and 5, we present finite element and time discretization, convergence analysis and the study of noise effects for the minimization problem. An a priori error estimate is then given, taking into account data noise, and a stopping criterion is proposed to control the instability of the minimization process. Finally, the numerical procedure and results are presented.

2. Statement of the problem

We consider a Lipschitz bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$ with n being the outward unit normal to the boundary $\Gamma = \partial\Omega$. Let us assume that Γ is partitioned into two parts, Γ_u (for *u*nknowns) and Γ_m (for *m*easurements), of the non-vanishing measurement, such that $\Gamma_u \cap \Gamma_m = \emptyset$. (Fig. 1.)

The most common problem consists in solving the heat transfer equation in a given domain Ω and a time interval $[0, D]$, assuming temperature distribution and heat flux are given over the accessible region of the boundary. We denote for $D > 0$

$$Q = \Omega\times]0, D[, \qquad \Sigma_u = \Gamma_u\times]0, D[, \qquad \Sigma_m = \Gamma_m\times]0, D[.$$

Given an initial temperature u_0 in Ω, a source term f, a conductivity field $\tilde{k}$, a density ρ and a heat capacity c in Q, a flux ϕ and the corresponding temperature T on Σ_m, the aim is to identify the corresponding flux and temperature on Σ_u. The nondimensionalized Cauchy problem is then written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla\cdot\big(k(x)\nabla u\big) = f \quad \text{in } Q \\ k(x)\nabla u\cdot n = \phi \quad \text{on } \Sigma_m \\ u = T \quad \text{on } \Sigma_m \\ u(\cdot, 0) = u_0 \quad \text{in } \Omega \end{cases} \tag{1}$$

where $k(x) = \tilde{k}(x)/\rho c$.

A problem is well-posed according to Hadamard (see [1,18,4]) if it fulfills the following properties: the uniqueness, existence and stability of the solution. The extended Holmgren theorem relating to Sobolev spaces (see [18]) guarantees uniqueness under regularity assumptions for the solution of the Cauchy problem. Since the well known Cauchy–Kowalevsky theorem (see [19]) is applicable only in the case of analytical data, the existence of this solution therefore depends on the verification of a compatibility condition difficult to formulate explicitly. In addition to the fact that for one fixed datum, the set of compatible data is dense within the full set of data (see [20]), this compatibility condition implies that the stability assumption is not satisfied in the sense that the dependence of a solution u of (1) on data (ϕ, T) is not continuous. Hereafter, we assume that data (ϕ, T) in (1) are compatible.

Notations: Let x be a generic point of Ω. The space of squared integrable functions $L^2(\Omega)$ is endowed with a natural inner product written as $(\cdot,\cdot)_{0,\Omega}$. The associated norm is written as $\|\cdot\|_{0,\Omega}$. We note $H^p(\Omega)$ the Sobolev space of functions of $L^2(\Omega)$ for which their pth order and lower derivatives are also in $L^2(\Omega)$. Its norm and seminorm are written as $\|\cdot\|_{p,\Omega}$ and $|\cdot|_{p,\Omega}$ respectively. Moreover, let $u = (u_1, u_2) \in (H^p(\Omega))^2$, the semi-norm of this space is written $|||u|||_{p,\Omega} = \big(|u_1|^2_{p,\Omega} + |u_2|^2_{p,\Omega}\big)^{1/2}$. Let $\gamma \subset \Gamma$, we define the space $H^1_{0,\gamma}(\Omega) = \{v \in H^1(\Omega);\, v_{|\gamma} = 0\}$ and $H^{1/2}_{00}(\gamma)$ is the space of restrictions to γ of the functions of $H^{1/2}(\Omega) = tr\big(H^1(\Omega)\big)$. Its topological dual is written as $H^{-1/2}_{00}(\gamma) = \big(H^{1/2}_{00}(\gamma)\big)'$. The associated norms are written as $\|\cdot\|_{1/2,00,\gamma}$ and $\|\cdot\|_{-1/2,00,\gamma}$, respectively and $\langle\cdot,\cdot\rangle_{1/2,00,\gamma}$ states for the duality inner product. Now, let t be the time variable. We denote by $L^2(0, D; F)$ the space of squared integrable functions in $[0, D]$ with values in F, where F is a normed functional space. In the same way, $\mathscr{C}^n(0, D; F)$ defines the space of n times continuously derivable functions in $[0, D]$ with values in F. The space of distributions in $]0, D[$ is written as $\mathscr{D}'(]0, D[)$. In the following C, indicates a positive generic constant.