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High-order compact finite difference schemes for option pricing in stochastic volatility models on non-uniform grids

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ABSTRACT

We derive high-order compact finite difference schemes for option pricing in stochastic volatility models on non-uniform grids. The schemes are fourth-order accurate in space and second-order accurate in time for vanishing correlation. In our numerical study we obtain high-order numerical convergence also for non-zero correlation and non-smooth payoffs which are typical in option pricing. In all numerical experiments a comparative standard second-order discretisation is significantly outperformed. We conduct a numerical stability study which indicates unconditional stability of the scheme.

1. Introduction

Efficient pricing of financial derivatives, in particular options, is one of the major topics in financial mathematics. To be able to explain important effects which are present in real financial markets, e.g. the volatility smile (or skew) in option prices, so-called *stochastic volatility* models have been introduced over the last two decades. In contrast to the seminal paper of Black and Scholes [1] the underlying asset's volatility is not assumed to be constant, but is itself modelled by a stochastic diffusion process. These stochastic volatility models are typically based on a two-dimensional stochastic diffusion process with two Brownian motions with correlation ρ, i.e. $dW^{(1)}(t)dW^{(2)}(t) = \rho\, dt$. On a given filtered probability space for the stock price $S = S(t)$ and the stochastic volatility $\sigma = \sigma(t)$ one considers

$$dS(t) = \bar{\mu}S(t)\,dt + \sqrt{\sigma(t)}S(t)\,dW^{(1)}(t),$$
$$d\sigma(t) = a(\sigma(t))\,dt + b(\sigma(t))\,dW^{(2)}(t),$$

where $\bar{\mu}$ is the drift of the stock, $a(\sigma)$ and $b(\sigma)$ are the drift and the diffusion coefficient of the stochastic volatility.

Application of Itô's Lemma and standard arbitrage arguments show that any option price $V = V(S, \sigma, t)$ solves the following partial differential equation,

$$V_t + \frac{1}{2}\sigma S^2\sigma V_{SS} + \rho b(\sigma)\sqrt{\sigma}SV_{S\sigma} + \frac{1}{2}b^2(\sigma)V_{\sigma\sigma} + \big(a(\sigma) - \lambda(S, \sigma, t)\big)V_\sigma + rSV_S - rV = 0, \tag{1}$$

where r is the (constant) riskless interest rate and $\lambda(S, \sigma, t)$ denotes the market price of volatility risk. Eq. (1) has to be solved for $S, \sigma > 0,\ 0 \le t \le T$, and subject to final and boundary conditions which depend on the specific option that is to be priced.

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There are different stochastic volatility models with different choices of the model for the evolution of the volatility for $t > 0$, starting from an initial volatility $\sigma(0) > 0$. The most prominent work in this direction is the Heston model [2], where

$$d\sigma(t) = \kappa^*\big(\theta^* - \sigma(t)\big)\,dt + v\sqrt{\sigma(t)}\,dW^{(2)}(t). \tag{2}$$

Other stochastic volatility models are, e.g., the GARCH diffusion model [3],

$$d\sigma(t) = \kappa^*\big(\theta^* - \sigma(t)\big)\,dt + v\sigma(t)\,dW^{(2)}(t), \tag{3}$$

or the so-called 3/2-model (see, e.g. [4]),

$$d\sigma(t) = \kappa^*\sigma(t)\big(\theta^* - \sigma(t)\big)\,dt + v\sigma(t)^{3/2}\,dW^{(2)}(t). \tag{4}$$

In (2)–(4), κ^*, v, and θ^* denote the mean reversion speed, the volatility of volatility, and the long-run mean of σ, respectively.

For some models and under additional restrictions, closed form solutions to (1) can be obtained by Fourier methods (see, e.g. [2,5]). Another approach is to derive approximate analytic expressions, see, e.g. [6] and the literature cited therein. In general, however—even in the Heston model when the parameters are non constant—Eq. (1) has to be solved numerically. Moreover, many (so-called American) options feature an additional early exercise right. Then one has to solve a free boundary problem which consists of (1) and an early exercise constraint for the option price. Also for this problem one typically has to resort to numerical approximations.

In the mathematical literature, there are a number of papers considering numerical methods for option pricing in stochastic volatility models, i.e. for two spatial dimensions. Finite difference approaches that are used are often standard, low order methods (second order in space). Other approaches include finite element-finite volume [7], multigrid [8], sparse wavelet [9], or spectral methods [10].

Let us review some of the related finite difference literature. Different efficient methods for solving the American option pricing problem for the Heston model are compared in [11]. The article focusses on the treatment of the early exercise free boundary and uses a second order finite difference discretisation. In [12] different, low order ADI (alternating direction implicit) schemes are adapted to the Heston model to include the mixed spatial derivative term. While most of [13] focusses on high-order compact scheme for the standard (one-dimensional) case, in a short remark [13, Section 5] also the stochastic volatility (two-dimensional) case is considered. However, the final scheme is of second order only due to the low order approximation of the cross diffusion term.

High-order finite difference schemes (fourth order in space) were proposed for option pricing with deterministic (or constant) volatility, i.e. in one spatial dimension, that use a compact stencil (three points in space), see, e.g., [13] for linear and [14–16] for fully nonlinear problems.

More recently, a high-order compact finite difference scheme for (two-dimensional) option pricing models with *stochastic volatility* has been presented in [17]. This scheme uses a uniform mesh and is fourth order accurate in space and second order accurate in time. Unconditional (von Neumann) stability of the scheme is proved for vanishing correlation. A further study of its stability, indicating unconditional stability also for non-zero correlation, is performed in [18].

In general, the accuracy of a numerical discretisation of (1) for a given number of grid points can be greatly improved by considering a *non-uniform mesh*. This is particular true for option pricing problems as (1), as typical initial conditions have a discontinuity in their first derivative at $S = K$, which is the centre of the area of interest ('at-the-money').

Our aim in the present paper is to consider extensions of the high-order compact methodology for stochastic volatility models (1) to non-uniform grids. The basic idea of our approach is to introduce a transformation of the partial differential equation from a non-uniform grid to a uniform grid (as, e.g. in [19]). Then, the high-order compact methodology can be applied to this transformed partial differential equation. It turns out, however, that this process is not straight-forward as the derivatives of the transformation appear in the truncation error and due to the presence of the cross-derivative terms, one cannot proceed to cancel terms in the truncation error in a similar fashion as in [17] and the derivation of a high-order compact scheme becomes much more involved. Nonetheless, we are able to derive a compact scheme which shows high-order convergence for typical European option pricing problems. Up to the knowledge of the authors, this is the first high-order compact scheme for option pricing in stochastic volatility models on non-uniform grids.

The rest of this paper is organised as follows. In the next section, we transform (1) into a more convenient form. We then derive four new variants of a compact scheme in Section 3. Numerical experiments confirming the high-order convergence for different initial conditions (we consider the case of a European Put option and a European Power Put option) are presented in Section 5. Section 6 concludes.

2. Transformation of the partial differential equation and final condition

We focus our attention on the Heston model (1)–(2), although our methodology adapts also to other stochastic volatility models in a natural way (see Remark 2 at the end of Section 3). As usual, we restrict ourselves to the case where the market price of volatility risk $\lambda(S, \sigma, t)$ is proportional to σ and choose $\lambda(S, \sigma, t) = \lambda_0\sigma$ for some constant λ_0. This allows to study