Journal of Computational and Applied Mathematics 263 (2014) 32–44

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Linear/linear rational spline collocation for linear boundary value problems

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ARTICLE INFO

Article history:
Received 23 March 2012
Received in revised form 22 November 2013

MSC:
65D05
65D07
65L10

Keywords:
Boundary value problems
Collocation
Rational spline
Convergence

ABSTRACT

We investigate the collocation method with linear/linear rational spline S of smoothness class C^1 for the numerical solution of two-point boundary value problems if the solution y of the boundary value problem is a strictly monotone function. We show that for the linear/linear rational splines on a uniform mesh it holds $\|S - y\|_\infty = O(h^2)$. Established bound of error for the collocation method gives a dependence on the solution of the boundary value problem and its coefficients. We prove also convergence rates $\|S' - y'\|_\infty = O(h^2)$, $\|S'' - y''\|_\infty = O(h)$ and the superconvergence of order h^2 for the second derivative of S in certain points. Numerical examples support the obtained theoretical results.

1. Introduction

Traditional methods for an approximate solution of boundary value problems are the finite difference method which only gives a discrete solution, and the collocation method with polynomial splines. The convergence rate $O(h^2)$ for quadratic and cubic spline collocation methods is known [1,2]. In some cases, the actual error is less for the quadratic splines and in other cases, the error is less for the cubic splines, see [2]. On the other hand, in interpolation the linear/linear rational splines of class C^1 have the same accuracy as the classical quadratic splines and none of them has advantage in comparison to real errors [3,4].

In such circumstances, it is natural to pose the question about relation between convergence rates of quadratic and linear/linear spline collocation for boundary value problems. In collocation with quadratic spline S it is known that $\|S - y\|_\infty = O(h^2)$, where y is the solution of the problem. For the proof see, e.g., [1,2]. We will study such a problem in the case of linear/linear rational spline. Let us mention that $O(h^2)$ convergence rate of quadratic spline collocation for boundary value problems is based on superconvergence property of interpolating splines. This was discovered in [1] and developed extensively in [5,2]. It was shown in [2] that the main part of the error at quadratic spline collocation is actually several times less than was obtained in [1]. For a smooth function y and interpolating linear/linear rational spline S it is known that $\|S - y\|_\infty = O(h^3)$, see, e.g., [3,6]. In [3], the expansions on subintervals via the derivatives of the smooth function to interpolate could be found as well. They give the superconvergence of the spline values and its derivatives in certain points.

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0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.11.028

Quite general results about stability and convergence of collocation methods with polynomials for boundary value problems are obtained in [7,8]. Piecewise polynomials are used as an approximate solution for the same problem in [9]. Cubic spline approximation at the solution of the two point boundary value problem for a second order linear differential equation is treated in [10–12] and also for fourth order boundary value problems in [13]. Collocation with some higher order splines for the second order boundary value problems is studied in [14]. Some surveys about other numerical methods for boundary value problems in ordinary differential equations could be found in [15–17]. In [18] collocation by piecewise Hermite cubic polynomials is made for two point boundary value problem and uniform convergence together with some superconvergence for the derivatives is obtained. These problems are dealt with in several books about spline theory [19,20] and numerical solution of differential equations. Among others we refer the reader to [21] for thorough treatment and comprehensive bibliography. Main ideas about the methods of solving nonlinear systems of equations could be found in [22].

In [23] *B*-spline collocation methods for the solution of initial value or boundary value problems are studied. The idea of transferring some initial conditions to final conditions is elaborated to recover the convergence lost in initial value methods. The appropriate analysis is carried out in [23,24]. The numerical solution of these problems using general non-uniform meshes is considered in [24,25]. Let us add that such an idea was successfully used for solving nonlinear Volterra integral equations by spline collocation, see [26,27]. The strategy, introduced in [25], of using quasi-interpolation for studying boundary value methods is considered in [28] as a general Hermite spline quasi-interpolation scheme and its convergence estimates are established.

Note that the linear/linear rational spline being constant or strictly monotone by itself is a reasonable approximate solution only if the exact solution of the problem has the same property. Let us point out that, while the collocation problem is a linear one, the linear/linear rational spline collocation is, in nature, a nonlinear method because it leads to a nonlinear system with respect to the spline parameters. However, the case of strictly monotone solution is restrictive. In opposite situation an adaptive strategy could be used. The theory of adaptive interpolation is developed, e.g., in [29] with cubic polynomial and quadratic/linear rational splines and in [30] for any data with quadratic polynomial and linear/linear rational splines. It is remarkable that the existence of those adaptive interpolating splines was quite complicated to prove. At spline collocation for boundary value problems this adaptive strategy needs to be investigated in future. It is natural that this research can be based on the results about rational spline collocation. Keeping this in mind, the main purpose of our paper is to show the existence of linear/linear rational spline as an approximate solution in the collocation method for boundary value problems and also comparison of error estimates with respect to quadratic spline collocation.

The considered boundary value problem and the collocation method with linear/linear rational splines are introduced in Section 2. Theorem 1 which is the main purpose of this paper and states the convergence and superconvergence estimates is also given at the end of Section 2. The next three sections are dedicated to the proof of Theorem 1, more precisely the necessary transformations are established in Section 3 and the use of the Bohl–Brouwer fixed point theorem is shown in Section 4. In Section 5 the convergence estimates are finally established. Numerical examples support the obtained theoretical results that can be found in the last section of this paper.

2. Collocation method with linear/linear rational splines

We consider the following boundary value problem

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \quad x \in (a, b), \tag{2.1}$$

with boundary conditions

$$y(a) = \alpha, \qquad y(b) = \beta.$$

Suppose that the problem has a solution $y \in C^4[a, b]$. Let q, f be continuous, p continuously differentiable and $q(x) \le q < 0$, $x \in (a, b)$. Then it is possible to show that the solution of problem (2.1) is unique.

Let $a = x_0 < x_1 < \cdots < x_n = b$ be a uniform partition of the interval $[a, b]$ with knots $x_i = a + ih,\ i = 0, \ldots, n,\ h = (b - a)/n,\ n \in \mathbb{N}$. We also need the points $\xi_i = x_{i-1} + h/2, i = 1, \ldots, n$. Linear/linear rational spline on each particular interval $[x_{i-1}, x_i]$ is a function S of the form

$$S(x) = a_i + \frac{c_i(x - \xi_i)}{1 + d_i(x - \xi_i)}, \quad x \in [x_{i-1}, x_i], \tag{2.2}$$

where $1 + d_i(x - \xi_i) > 0$.

As C^1 linear/linear rational spline is strictly increasing or strictly decreasing or constant on $[a, b]$, we assume that $y'(x) > 0$ for all $x \in [a, b]$.

Using the notation $S(x_i) = S_i,\ i = 0, \ldots, n$, and $S(\xi_i) = \bar{S}_i,\ i = 1, \ldots, n$, we get from (2.2)

$$\bar{S}_i = a_i,$$

$$S_{i-1} = a_i - \frac{hc_i}{2 - hd_i},$$

$$S_i = a_i + \frac{hc_i}{2 + hd_i}$$