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A multivariate dependence measure for aggregating risks

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ABSTRACT

To evaluate the aggregate risk in a financial or insurance portfolio, a risk analyst has to calculate the distribution function of a sum of random variables. As the individual risk factors are often positively dependent, the classical convolution technique will not be sufficient. On the other hand, assuming a comonotonic dependence structure will likely overrate the real aggregate risk. In order to choose between the two approximations, or perhaps use a weighted average, we should have an indication of the accuracy. Clearly this accuracy will depend on the copula between the individual risk factors, but it is also influenced by the marginal distributions. In this paper we introduce a multivariate dependence measure that takes both aspects into account. This new measure differs from other multivariate dependence measures, as it focuses on the aggregate risk rather than on the copula or the joint distribution function itself. We prove several interesting properties of this new measure and discuss its relation to other dependence measures. We also give some comments on the estimation and conclude with examples and numerical results.

1. Introduction

When evaluating the risk exposure of a financial or insurance portfolio, the risk analyst has to evaluate a sum of random variables. Consider a portfolio $\boldsymbol{X}$ consisting of d risk factors $X_1, X_2, \ldots, X_d$; then the aggregate risk is $S = X_1 + X_2 + \cdots + X_d$. To determine the distribution of this aggregate risk, we have to know the joint distribution $F_{\boldsymbol{X}}$ of $(X_1, X_2, \ldots, X_d)$. In practice however this often turns out to be very difficult. Modeling the marginal distributions of X_i is quite a common task, but finding the appropriate copula between the X_i is much less straightforward. Moreover the calculation of the aggregate distribution involves a d-dimensional integral, which is not very appealing for high-dimensional portfolios.

One way to tackle this problem is to simply neglect the dependence and assume that the risks are independent. Let $\boldsymbol{X}^\perp = (X_1^\perp, X_2^\perp, \ldots, X_d^\perp)$ be a random vector with the same marginal distributions as $\boldsymbol{X}$ but with independent components, i.e. $\boldsymbol{X}^\perp$ has cumulative distribution

$$F_{\boldsymbol{X}^\perp}(\boldsymbol{x}) = F_{X_1}(x_1) F_{X_2}(x_2) \cdots F_{X_d}(x_d).$$

The distribution of $S^\perp = X_1^\perp + X_2^\perp + \cdots + X_d^\perp$ can be obtained by the well-known convolution technique or, for some specific marginal distributions, by using the recursion formulas given in [1] and other works. Obviously, neglecting the (usually positive) dependence, we might underrate the aggregate risk as $S^\perp$ will usually have a smaller variance. Note however that $\mathrm{E}[S] = \mathrm{E}[S^\perp]$, as $\boldsymbol{X}$ and $\boldsymbol{X}^\perp$ belong to the same Fréchet class.

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Alternatively, one might consider the strongest positive dependence and assume that the risks are comonotonic. Let $\boldsymbol{X}^c = (X_1^c, X_2^c, \ldots, X_d^c)$ be a random vector with the same marginal distributions as $\boldsymbol{X}$ but with comonotonic components, i.e. $\boldsymbol{X}^c$ has cumulative distribution

$$F_{\boldsymbol{X}^c}(\boldsymbol{x}) = \min\{F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_d}(x_d)\}$$

or, equivalently,

$$\boldsymbol{X}^c =_d (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_d}^{-1}(U)), \quad U \sim U(0, 1)$$

where $=_d$ denotes equality in distribution. The distribution function of $S^c = X_1^c + X_2^c + \cdots + X_d^c$ can be obtained by inverting the quantile function, which in turn equals the sum of the marginal quantile functions $F_{X_i}^{-1}$. Dhaene et al. [2] show that S is smaller in convex order than S^c (written $S \leq_{cx} S^c$), i.e.

$$\mathrm{E}[v(S)] \leq \mathrm{E}[v(S^c)]$$

for all real convex functions v, provided the expectations exist. This implies that S^c has heavier tails than S and $\mathrm{Var}[S^c] \geq \mathrm{Var}[S]$, so the aggregate risk will likely be overrated. Note that $\boldsymbol{X}$ and $\boldsymbol{X}^c$ also belong to the same Fréchet class, so $\mathrm{E}[S] = \mathrm{E}[S^c]$.

In order to choose between the two approximations, or perhaps use a weighted average, we should have an indication of the accuracy. Clearly this accuracy will depend on the copula of $\boldsymbol{X}$, but it is also influenced by the marginal distributions. In this paper we introduce a multivariate dependence measure that takes both aspects into account. This new measure differs from other multivariate dependence measures in e.g. [3–7] or [8], as it focuses on the aggregate risk S rather than on the copula or the joint distribution function of $\boldsymbol{X}$. In a finance context, it can be translated into a measure for herd behavior; see [9].

In the following section we introduce the new multivariate dependence measure and prove several interesting properties. We also discuss its relation to the classical Pearson correlation and the comonotonicity coefficient of Koch and De Schepper [8]. In Section 3 we give some comments on the estimation and Section 4 concludes with examples and numerical results.

2. The definition and properties

Most of the multivariate dependence measures proposed in the literature are written directly in terms of the copula or the joint distribution function of $\boldsymbol{X}$. Keeping the aggregate risk in mind, we propose to measure the dependence in $\boldsymbol{X}$ indirectly through the distribution of the sum S of its components. More specifically, we will focus on the variance of S. As convex order implies ordered variances, we have that $\mathrm{Var}(S) \leq \mathrm{Var}(S^c)$. This suggests the following multivariate dependence measure.

Definition 2.1. The dependence measure ρ_c of a random vector $\boldsymbol{X}$ with non-degenerate margins is defined as

$$\rho_c(\boldsymbol{X}) = \frac{\mathrm{Var}(S) - \mathrm{Var}(S^\perp)}{\mathrm{Var}(S^c) - \mathrm{Var}(S^\perp)} = \frac{\sum_{i=1}^{d}\sum_{j<i}\mathrm{Cov}(X_i, X_j)}{\sum_{i=1}^{d}\sum_{j<i}\mathrm{Cov}(X_i^c, X_j^c)} \tag{1}$$

provided the covariances exist.

The first expression in (1) has a similar structure to the multivariate dependence measures ρ_n in [3] and κ in [8]. Both measures are also centered around the independent vector and normalized with respect to the comonotonic vector. From the second expression we see that ρ_c can be interpreted as a normalized average of bivariate covariances. Since the numerator cannot exceed the denominator, ρ_c is bounded from above by 1; see Proposition 2.8. Without imposing some restrictions on $\boldsymbol{X}$ however, there is no general lower bound; see Proposition 2.9.

The condition of non-degenerate margins ensures that the denominator in (1) is non-zero. Before we prove this assertion, we extend a result of Luan [10] for positive random variables to real-valued random variables. The proof in [10] relies on the assumption that X and Y are bounded from below, so we give a somewhat different proof.

Lemma 2.2. *Two random variables X and Y are both independent and comonotonic if and only if at least one of them is degenerate.*

Proof. First, assume that Y is degenerate with value a, i.e. $\mathrm{P}(Y = a) = 1$ and $\mathrm{P}(Y \neq a) = 0$. Then,

$$F_{X,Y}(x, y) = \mathrm{P}(X \leq x, Y \leq y) = \begin{cases} 0 & y < a \\ F_X(x) & y \geq a \end{cases}$$

and

$$\min\left(F_X(x), F_Y(y)\right) = F_X(x)F_Y(y) = \begin{cases} 0 & y < a \\ F_X(x) & y \geq a \end{cases}$$

so X and Y are both independent and comonotonic.