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Optimality of adaptive Galerkin methods for random parabolic partial differential equations

Claude Jeffrey Gittelson [a], Roman Andreev [b,*], Christoph Schwab [c]

[a] Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA
[b] RICAM, Austrian Academy of Sciences, Altenberger-Str. 69, 4040 Linz, Austria
[c] Seminar for Applied Mathematics, ETH Zürich, Rämistrasse 101, CH-8092 Zürich, Switzerland

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ABSTRACT

Galerkin discretizations of a class of parametric and random parabolic partial differential equations (PDEs) are considered. The parabolic PDEs are assumed to depend on a vector $y = (y_1, y_2, \ldots)$ of possibly countably many parameters y_j which are assumed to take values in $[-1, 1]$. Well-posedness of weak formulations of these parametric equations in suitable Bochner spaces is established. Adaptive Galerkin discretizations of the equation based on a tensor product of a generalized polynomial chaos in the parameter domain $\Gamma = [-1, 1]^{\mathbb{N}}$, and of suitable wavelet bases in the time interval $I = [0, T]$ and the spatial domain $D \subset \mathbb{R}^d$ are proposed and their optimality is established.

0. Introduction

In recent years, based on the pioneering works [1,2], and the subsequent refinements [3–7], a rigorous theory of optimal (in the sense that convergence rates which are afforded by best N-term approximations from a biorthogonal expansion of the unknown solution in some *a priori* given Riesz basis are achieved) adaptive Galerkin approximation methods has emerged. After initial applications to linear elliptic partial differential equations in [1,2] using isotropically supported multiresolution bases, extensions to integro-differential operators have been considered in [3,4], first applications to elliptic multiscale problems using anisotropic tensor product Riesz bases have been considered in [5,6] and, subsequently, to the space–time compressive discretization of linear parabolic (integro)differential equations have been considered in [7].

* Corresponding author.
E-mail addresses: cgittels@purdue.edu (C.J. Gittelson), roman.andreev@oeaw.ac.at (R. Andreev), schwab@sam.math.ethz.ch (C. Schwab).

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In recent years, in particular in connection with the numerical solution of partial differential equations with random inputs, for example with random coefficients given by Karhunen–Loève expansions, initial boundary value problems of parametric, deterministic partial differential operators which depend on a sequence of countably many parameters have been considered. Various discretization approaches, for example collocation and Monte Carlo sampling techniques, have been considered (see, *e.g.*, [8] and the references therein).

While affording convenient implementation, the analysis of sampling methods currently leaves open the question of optimality. Here, the situation for the so-called stochastic Galerkin discretizations is quite different: since the discretization consists in a mean-square projection onto a polynomial chaos, *i.e.* onto a finite span from a countable ensemble of tensorized orthogonal polynomials, in principle techniques for establishing optimality of Galerkin projection methods for the approximate solution of operator equations can be brought to bear. This programme has been implemented in [9] and the references therein for parametric operator equations.

In the present paper, we adapt these techniques to prove optimality of an adaptive Galerkin scheme for linear, parametric and parabolic equations. Here, we use a Legendre generalized polynomial chaos in the parameter space, and a space–time tensor product wavelet basis that was shown to lead to an optimal Galerkin approximation for the non-parametric, parabolic initial boundary problems in [7]. Based on the approach and the tensorized space–time Riesz bases for the Bochner space in these references, we develop in the present paper a family of adaptive Galerkin discretizations which are based on tensorizing the generalized polynomial chaos and the space–time tensor product wavelet bases, resulting in discretization schemes which are simultaneously adaptive in space–time and in the parameter space. We establish here optimality of the resulting algorithm, which implies that the best N-term approximation rates which are afforded by the exact solution from the tensorized basis are, indeed, realized by the sequence of finitely supported approximations generated by the proposed adaptive Galerkin discretization.

The outline of this paper is as follows. In Section 1.1, we present an abstract class of parametric, parabolic problems which may depend on a countable number of parameters. We elaborate on the specific class of *affine parameter dependence* of the parametric operator equations.

In Section 2, we introduce a space–time weak formulation which also includes a weak form of the parameter dependence.

Sections 3 and 4 introduce the requirement for polynomial chaos type Riesz bases in the parameter domain, and for the multiresolution (wavelet) Riesz bases on the space and time domains.

Section 5 introduces an equivalent bi-finite matrix equation which, in particular, allows for suitable compressibility results.

Section 6 presents elements of the general adaptive Galerkin framework, based on the general Refs. [1,2,4] where adaptive wavelet methods were developed in the context of wavelet discretizations of elliptic operator equations, to the extent required by the ensuing developments.

Section 7 recapitulates from [9] general results on the optimality of adaptive Galerkin approximations of deterministic operator equations. Finally, Section 8 contains statements and proofs of the main result of the present paper, the optimality of the proposed adaptive Galerkin approximations in space, time and parameter domains by sparse, tensorized bases consisting of tensor products of Riesz bases $\boldsymbol{\Theta}$, $\boldsymbol{\Sigma}$ and of $\boldsymbol{P}$.

1. Random and parametric parabolic equations

1.1. Abstract setting

Let V and H be real or complex separable Hilbert spaces. We denote by V^* the dual space of V, which consists of all bounded antilinear functionals on V. Assuming a dense embedding $V \hookrightarrow H$, we obtain a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$, where H is canonically identified with its dual.

We shall consider equations in V that depend on a temporal variable $t \in I := [0, T]$ and also on a parameter sequence $y \in \Gamma := [-1, 1]^{\mathbb{N}}$. On Γ, we define a probability measure

$$\boldsymbol{\pi} = \bigotimes_{m\in\mathbb{N}} \pi_m, \tag{1.1}$$

where each π_m is assumed to be a probability measure on $[-1, 1]$ with the Borel σ-algebra. Although the product structure of the domain Γ and the measure π is irrelevant for the abstract problem formulation, it is pivotal to the subsequent construction of countable orthonormal (with respect to $\boldsymbol{\pi}$) bases on the parameter domain in Section 3.

For a.e. $t \in I$ and π-a.e. $y \in \Gamma$, we denote by $A(t, y; \cdot, \cdot)$ a sesquilinear form on $V \times V$ such that for any $v, w \in V$, the map $(t, y) \mapsto A(t, y; v, w)$ is Borel-measurable on $I \times \Gamma$, and such that for a.e. t and y

$$|A(t, y; v, w)| \le c_{\max} \|v\|_V \|w\|_V \quad \forall v, w \in V, \tag{1.2}$$

$$\Re A(t, y; v, v) + c_0 \|v\|_H^2 \ge c_{\min} \|v\|_V^2 \quad \forall v \in V, \tag{1.3}$$

with fixed constants $c_{\max} > 0$, $c_{\min} > 0$ and $c_0 \ge 0$. For any $v \in V$, the antilinear functional $A(t, y; v, \cdot)$ is an element of V^*. This allows us to interpret $A(t, y)$ as a bounded linear map from V to V^* for a.e. t and for $\boldsymbol{\pi}$-a.e. $y \in \Gamma$.