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Preserving convexity through rational cubic spline fractal interpolation function

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ABSTRACT

We propose a new type of $\mathcal{C}^1$-rational cubic spline Fractal Interpolation Function (FIF) for convexity preserving univariate interpolation. The associated Iterated Function System (IFS) involves rational functions of the form $\frac{P_n(x)}{Q_n(x)}$, where $P_n(x)$ are cubic polynomials determined through the Hermite interpolation conditions of the FIF and $Q_n(x)$ are preassigned quadratic polynomials with two shape parameters. The rational cubic spline FIF converges to the original function Φ as rapidly as the rth power of the mesh norm approaches to zero, provided $\Phi^{(r)}$ is continuous for $r = 1$ or 2 and certain mild conditions on the scaling factors are imposed. Furthermore, suitable values for the rational IFS parameters are identified so that the property of convexity carries from the data set to the rational cubic FIFs. In contrast to the classical non-recursive convexity preserving interpolation schemes, the present fractal scheme is well suited for the approximation of a convex function Φ whose derivative is continuous but has varying irregularity.

1. Background and preliminaries

Suppose a set of data points $\mathcal{D} = \{(x_n, y_n) \in I \times \mathbb{R} : n = 1, 2, \ldots, N\}$ is given, where $x_1 < x_2 < \cdots < x_N$ and $I = [x_1, x_N]$. The problem of interpolation in numerical analysis and approximation theory deals with the construction of a continuous function $S : I \to \mathbb{R}$ satisfying $S(x_n) = y_n$ for $n = 1, 2, \ldots, N$. The interpolants produced by these traditional methods are smooth, sometimes infinitely (piecewise) differentiable. As a consequence, these methods become unsuitable for interpolating a highly irregular data or approximating a function whose derivative of a certain order is irregular in a dense subset of the interpolation interval. This served as a motivation for the development of a new interpolation technique using fractal methodology.

The theory of fractals and fractal interpolation functions has evolved beyond its mathematical framework and has become a powerful tool in the applied sciences as well as engineering [1–3]. The realm of applications of fractals and FIFs includes but not limited to geometric design [4], structural mechanics [5], physics and chemistry [6,7], signal processing and decoding [8,9], and applied wavelet theory [10]. The reason for the vast applications of FIFs is attributable to their ability to produce complicated mathematical structures with a simple recursive procedure. A FIF is constructed as a fixed point of the Read–Bajraktarević operator defined on a suitable function space. In what follows, we shall recall the precise definition of a FIF and its construction proposed by Barnsley [2].

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Set $I_n = [x_n, x_{n+1}]$ for $n \in J = \{1, 2, \dots, N-1\}$. Let $L_n : I \to I_n,\ n \in J$, be contraction homeomorphisms such that:

$$L_n(x_1) = x_n, \qquad L_n(x_N) = x_{n+1}. \tag{1.1}$$

Let $-1 < \alpha_n < 1,\ n \in J$. Further, let $K = I \times [a, b]$ for some $-\infty < a < b < +\infty$, and $N-1$ continuous mappings $F_n : K \to [a, b]$ be given satisfying:

$$\begin{aligned} &F_n(x_1, y_1) = y_n, \qquad F_n(x_N, y_N) = y_{n+1}, \quad n \in J, \\ &|F_n(x, y_l) - F_n(x, y_m)| \le |\alpha_n||y_l - y_m|, \quad x \in I,\ y_l, y_m \in [a, b]. \end{aligned} \tag{1.2}$$

Define functions $w_n : K \to K,\ w_n(x, y) = (L_n(x), F_n(x, y))\ \forall\, n \in J$. Consider the collection $\mathcal{I} \equiv \{K;\ w_n : n \in J\}$, which is termed as an IFS. Associated with the collection of functions in $\mathcal{I}$, there is a set valued mapping w from the hyperspace $\mathcal{H}(K)$ of nonempty compact subsets of K into itself. More precisely, $w(A) = \cup_{n\in J} w_n(A)$ for $A \in \mathcal{H}(K)$, where $w_n(A) = \{w_n(a) : a \in A\}$. There exists a metric h, called the Hausdorff metric, which completes $\mathcal{H}(K)$. This metric is defined as $h(A, B) = \max\{\max_{x\in A} \min_{y\in B} d(x, y), \max_{y\in B} \min_{x\in A} d(y, x)\}\ \forall A, B \in \mathcal{H}(K)$. Here d is a metric that is equivalent to the Euclidean metric on $\mathbb{R}^2$ with respect to which each w_n is a contraction. It is well known [2] that w is a contraction on the complete metric space $(\mathcal{H}(K), h)$. Consequently, by the Banach Fixed Point Theorem there exists a unique set G such that $G = \lim_{n\to\infty} w^n(A_0)$ and $w(G) = G$, where $A_0 \in \mathcal{H}(K)$ is arbitrary. Here w^n denotes the n-fold composition of w, and the limit is taken in the Hausdorff metric: $w^n(A_0) \to G \Leftrightarrow h(G, w^n(A_0)) \to 0$. Such a set G is called an attractor or a deterministic fractal. The next proposition relates G with a function interpolating the data set.

Proposition 1.1 (*Barnsley [1]*). *The IFS $\{K; w_n : n \in J\}$ defined above admits a unique attractor G. Further, G is the graph of a continuous function $S : I \to \mathbb{R}$ which obeys $S(x_n) = y_n$ for $n = 1, 2, \dots, N$.*

The function S whose graph is the attractor of an IFS as described in Proposition 1.1 is called a FIF. Now we provide some excerpts from the proof of the above proposition that yield a functional equation corresponding to the interpolant S.

Let $\mathcal{G} := \{H : I \to \mathbb{R} \mid H \text{ is continuous}, H(x_1) = y_1 \text{ and } H(x_N) = y_N\}$. Then $\mathcal{G}$ endowed with the uniform metric $\tau(H, H^*) := \max\{|H(x) - H^*(x)| : x \in I\}$ is a complete metric space. Define the Read–Bajraktarević operator T on $(\mathcal{G}, \tau)$ as follows:

$$(TH)(x) = F_n\big(L_n^{-1}(x), H \circ L_n^{-1}(x)\big), \quad x \in I_n,\ n \in J. \tag{1.3}$$

Due to the conditions on the maps L_n and $F_n,\ n \in J$, it follows that TH is continuous on I. Furthermore, the map T is a contraction on the metric space $(\mathcal{G}, \tau)$, i.e., $\tau(TH, TH^*) \le |\alpha|_\infty \tau(H, H^*)$, where $|\alpha|_\infty := \max\{|\alpha_n| : n \in J\} < 1$. Since $(\mathcal{G}, \tau)$ is complete, T possesses a unique fixed point S, i.e., there is $S \in \mathcal{G}$ such that $(TS)(x) = S(x)\ \forall\, x \in I$. This function S is the FIF corresponding to the IFS $\mathcal{I}$. Therefore, from (1.3) it follows that S satisfies the functional equation:

$$S(x) = F_n\big(L_n^{-1}(x), S \circ L_n^{-1}(x)\big), \quad x \in I_n,\ n \in J. \tag{1.4}$$

The most extensively studied FIFs so far in the literature stem from the IFS

$$\big\{K;\ w_n(x, y) \equiv \big(L_n(x) = a_n x + b_n,\ F_n(x, y) = \alpha_n y + R_n(x)\big) : n \in J\big\}, \tag{1.5}$$

where $R_n : I \to \mathbb{R}$ are suitable continuous functions, generally polynomials, such that the conditions prescribed in (1.2) are satisfied. The parameter α_n is called a scaling factor of the transformation w_n, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$ is the scale vector of the IFS. The main differences of a FIF with the traditional interpolation techniques consist: (i) in the definition in terms of a functional equation that implies a self similarity in small scales, (ii) in the constructive way through iterations that is used to compute the interpolant instead of analytic formulae, (iii) in the presence of free parameters α_n that replace the unicity of the traditional interpolant for a fixed set of interpolation data with unicity of the interpolant for a fixed data set and a fixed choice of scale vector, and (iv) in the fact that the fractal dimension of a FIF is, in general, non-integer. To obtain the actual fractal interpolant, one needs to continue the iterations indefinitely. However, a small number of iterations usually yields a close approximation. Next we recall the following result of Barnsley and Harrington [11] for the construction of an IFS that generates smooth interpolants corresponding to a finite set of data points.

Proposition 1.2 (*Barnsley and Harrington [11]*). *Let $\{(x_n, y_n) : n = 1, 2, \dots, N\}$ be a given set of data points, where $x_1 < x_2 < \dots < x_N$. Let $L_n(x) = a_n x + b_n$, $n \in J$, be the affine functions satisfying (1.1) and $F_n(x, y) = \alpha_n y + R_n(x)$, $n \in J$, satisfy (1.2). Suppose that for some integer $p \ge 0$, $|\alpha_n| < a_n^p$ and $R_n \in \mathcal{C}^p[x_1, x_N]$, $n \in J$. Let*

$$F_{n,k}(x, y) = \frac{\alpha_n y + R_n^{(k)}(x)}{a_n^k}, \qquad y_{1,k} = \frac{R_1^{(k)}(x_1)}{a_1^k - \alpha_1}, \qquad y_{N,k} = \frac{R_{N-1}^{(k)}(x_N)}{a_{N-1}^k - \alpha_{N-1}}, \quad k = 1, 2, \dots, p.$$

If $F_{n-1,k}(x_N, y_{N,k}) = F_{n,k}(x_1, y_{1,k})$ for $n = 2, 3, \dots, N-1$ and $k = 1, 2, \dots, p$, then the IFS $\{I \times \mathbb{R}; (L_n(x), F_n(x, y)), n \in J\}$ determines a FIF $S \in \mathcal{C}^p[x_1, x_N]$, and $S^{(k)}$ is the FIF determined by $\{I \times \mathbb{R}; (L_n(x), F_{n,k}(x, y)), n \in J\}$ for $k = 1, 2, \dots, p$.