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A posteriori error estimation for the Lax–Wendroff finite difference scheme

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HIGHLIGHTS

- Formulation of the Lax–Wendroff difference scheme for conservation laws as a finite element method.
- Goal oriented a posteriori error estimate for the Lax–Wendroff finite difference scheme.
- Investigation of accuracy of the computational error estimate.

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ABSTRACT

In many application domains, the preferred approaches to the numerical solution of hyperbolic partial differential equations such as conservation laws are formulated as finite difference schemes. While finite difference schemes are amenable to physical interpretation, one disadvantage of finite difference formulations is that it is relatively difficult to derive the so-called goal oriented a posteriori error estimates. A posteriori error estimates provide a computational approach to numerically compute accurate estimates in the error in specified quantities computed from a numerical solution. Widely used for finite element approximations, a posteriori error estimates yield substantial benefits in terms of quantifying reliability of numerical simulations and efficient adaptive error control.

The chief difficulties in formulating a posteriori error estimates for finite difference schemes is introducing a variational formulation – and the associated adjoint problem – and a systematic definition of residual errors. In this paper, we approach this problem by first deriving an equivalency between a finite element method and the Lax–Wendroff finite volume method. We then obtain an adjoint based error representation formula for solutions obtained with this method. Results from linear and nonlinear viscous conservation laws are given.

1. Introduction

In this paper, we derive a computable, goal-specific a posteriori error estimate for the Lax–Wendroff finite difference scheme for the viscous nonlinear conservation law in one dimension,

$$\begin{cases} u_t + f(u)_x = \epsilon u_{xx}, & x \in \mathbb{S}^1, 0 < t \le T, \\ u(x,0) = u_0(x), & x \in \mathbb{S}^1, \end{cases} \tag{1}$$

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where $\epsilon > 0, f : \mathbb{R} \to \mathbb{R}$ is smooth, and $\mathbb{S}^1$ is the one dimensional unit sphere, i.e. we assume periodic boundary conditions. We also apply the estimate to an example with $\epsilon = 0$, in which case, we also assume f is convex. Periodic boundary conditions greatly simplifies the presentation since boundary conditions can introduce serious complications for hyperbolic and convection-dominated problems. Generally, a posteriori error estimates can be extended to include the effects of error in boundary conditions and pursuing such analysis for hyperbolic equations is an interesting problem.

In contrast to a priori convergence and accuracy analysis, a posteriori error estimate yields an accurate estimate of the error in information $\mathcal{Q}(u)$ computed from a particular numerical solution U. The ingredients of the a posteriori error analysis include variational analysis, adjoint operators, and computable residuals. Computable accurate error estimates are an important component of reliability, uncertainty quantification, and adaptive error control. Adjoint-based a posteriori error estimation has been developed and implemented widely over the past few decades within the finite element community [1–4]. Much of the work in a posteriori error estimation has been directed towards elliptic and parabolic problems, however there is some recent research targeting conservation laws. Barth and Larson, [5–7], considered error estimation for the discontinuous Galerkin method and certain Godunov methods. Other work, [8–11] has addressed adaptivity and the necessary error estimation for various conservation laws. All of the studies for conservation laws assume the approximate solution is obtained by a finite element method e.g., discontinuous Galerkin. This method is well-suited for a posteriori error estimation, but it is also relatively new.

The first methods developed for hyperbolic problems were finite difference methods. These included methods such as Lax–Wendroff [12], Godunov [13], MacCormack [14], upwind [15], and many others[16,17]. See [18,19] for a review of some of the early finite difference schemes for hyperbolic problems. These methods were developed to deal with hyperbolic problems, in particular, to capture discontinuities effectively. They were also developed to have low computational cost, being explicit methods. Therefore, many large scale codes implement these methods [20–22]. Therefore, it is useful to obtain a posteriori error estimates for solutions obtained by these finite difference methods.

In this paper, we derive an a posteriori error estimate for the Lax–Wendroff scheme. The main ideas of the analysis can be used to derive estimates for other finite difference schemes, though the specific details would depend on the particular scheme in question. To derive the estimate, we first rewrite the Lax–Wendroff method as a ''nodally equivalent'' finite element method. We then perform an adjoint based error analysis for this finite element method, which can then be interpreted as an estimate for the original difference scheme. The error estimate can be partitioned into a sum of contributions, each corresponding to specific approximations made in the discretization. This quantification of various contributions to the error is essential to obtain an accurate estimate and it is also useful for adaptivity, as discussed in the conclusion. Since this scheme is explicit, we use the work of [23], where error estimation was performed for explicit time stepping schemes for ordinary differential equations.

The structure of the paper is organized as follows. We recall the derivation of the Lax–Wendroff scheme in Section 2. In Section 3, we formulate a finite element method that is equivalent to the Lax–Wendroff scheme. We present the a posteriori error estimate in Section 4. Numerical results for the linear advection and Burgers equations are presented in Section 5.

2. A review of the Lax–Wendroff finite difference scheme

The Lax–Wendroff scheme is an explicit second order difference scheme. As with other simple difference schemes, simplicity of implementation is an attractive feature. However, the price of higher order approximation is that the Lax–Wendroff scheme is dispersive, which limits usefulness for problems with shocks. Nonetheless, it is still an extremely popular method that is embedded in many legacy codes.

We partition the temporal domain by the nodes, $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, and define $k_n = t_n - t_{n-1}$, while the spatial domain is partitioned by the nodes, $x_M = x_0 < x_1 < \cdots < x_{M-1} < x_M = x_0$, with the uniform spatial step $h = x_i - x_{i-1}$. The Lax–Wendroff scheme is originally derived for a pure convection problem, that is (1) with $\epsilon = 0$, based on a truncated Taylor series expansion. Assuming $u(x, t)$ is a smooth solution of (1) with $\epsilon = 0$, we consider the approximation of the solution generated by truncating the Taylor series in time:

$$u(x, t + k_n) \approx u(x, t) + k_n u_t(x, t) + \frac{k_n^2}{2} u_{tt}(x, t). \tag{2}$$

Then, using (1), we replace all temporal derivatives with spatial derivatives,

$$u_t = f(u)_x, \qquad u_{tt} = f(u)_{xt} = f(u)_{tx} = (f'(u)u_t)_x = (f'(u)f(u)_x)_x.$$

Approximating the spatial derivatives with centered differences, and using subscripts and superscripts to denote the finite difference approximation, we obtain the update formula for the Lax–Wendroff method,

$$u_i^n = u_i^{n-1} - \frac{k_n}{2h}(f_{i+1}^{n-1} - f_{i-1}^{n-1}) - \frac{k_n^2}{2h^2}\left[f_{i-1/2}^{\prime n-1}(f_i^{n-1} - f_{i-1}^{n-1}) + f_{i+1/2}^{\prime n-1}(f_i^{n-1} - f_{i+1}^{n-1})\right], \tag{3}$$

for $n = 1, \ldots, N$ and $i = 0, \ldots, M - 1$, and

$$u_i^n = u(x_i, t_n), \qquad f_i^n = f(u(x_i, t_n)), \qquad f_{i+1/2}^{\prime n} = f'(u(x_{i+1/2}, t_n)).$$

It is also common to approximate the propagation speed evaluated at the midpoint by, $f_{i+1/2}^{\prime n} \approx f'\left(\frac{u_i^n + u_{i+1}^n}{2}\right)$.