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A reweighted nuclear norm minimization algorithm for low rank matrix recovery☆

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ABSTRACT

In this paper, we propose a reweighted nuclear norm minimization algorithm based on the weighted fixed point method (RNNM–WFP algorithm) to recover a low rank matrix, which iteratively solves an unconstrained L_2–M_p minimization problem introduced as a nonconvex smooth approximation of the low rank matrix minimization problem. We prove that any accumulation point of the sequence generated by the RNNM–WFP algorithm is a stationary point of the L_2–M_p minimization problem. Numerical experiments on randomly generated matrix completion problems indicate that the proposed algorithm has better recoverability compared to existing iteratively reweighted algorithms.

1. Introduction

The low rank matrix minimization problem (LRM for short) is to find a lowest rank matrix based on some feasible measurement ensembles. When the set of measurement is affine in the matrix variable, the LRM is given by

$$\min_{X} \ \text{rank}(X) \quad \text{s.t. } \mathcal{A}(X) = b, \tag{1.1}$$

where the linear map $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^s$ and the vector $b \in \mathbb{R}^s$ are known.

The LRM has various applications in image compression, statistics embedding, system identification and control problems; and it is NP-hard. The tightest convex relaxation of problem (1.1) is the following nuclear norm minimization problem:

$$\min_{X} \ \|X\|_* \quad \text{s.t. } \mathcal{A}(X) = b, \tag{1.2}$$

where $\|X\|_* := \sum_{i=1}^{r} \sigma_i(X)$ is the sum of all the singular values of $X \in \mathbb{R}^{m\times n}$ with $\text{rank}(X) = r$ (here $n \leq m$ without loss of generality). When X is restricted to be a diagonal matrix, problems (1.1) and (1.2) reduce to the sparse signal recovery problem:

$$\min_{x} \ \|x\|_0 \quad \text{s.t. } Ax = b \tag{1.3}$$

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and the l_1 norm minimization problem:

$$\min_x \ \|x\|_1 \quad \text{s.t.}\, Ax = b, \tag{1.4}$$

respectively, where $\|x\|_0 := |\text{support}(x)| := |\{i \mid x_i \neq 0\}|$ is the cardinality of the support set of $x \in \mathbb{R}^n$, and $\|x\|_1 := \sum_{i=1}^n |x_i|$ is the l_1-norm of the vector x.

It is known that under some conditions on the linear transformation $\mathcal{A}$ (or matrix A), one can obtain an exact solution of problem (1.1) (or (1.3)) via (1.2) (or (1.4)) [1,2]. Besides, problem (1.1) (or (1.3)) can also be relaxed to the nonconvex optimization problem, such as, replacing the term $\|X\|_*$ in (1.2) by the nonconvex term $\|X\|_p^p := \sum_{i=1}^n \sigma_i^p(X)$ (or replacing $\|x\|_1$ in (1.4) by $\|x\|_p^p := \sum_{i=1}^n |x_i|^p$) with $0 < p < 1$. It was shown by Chartrand [3] that a nonconvex variant of (1.3) could produce exact reconstruction with fewer measurements. Recently, there has been an explosion of research on this topic both in vector case and matrix case, see, e.g., [4–14].

Recently, some nonconvex smooth approximation models were proposed in the literature. In the context of the vector case, one of the models is the following unconstrained l_2–l_p minimization problem:

$$\min_x \ \lambda \sum_{i=1}^n (|x_i| + \varepsilon)^p + \|Ax - b\|_2^2 \quad \text{where } 0 < p \le 1. \tag{1.5}$$

This problem can be solved by iteratively reweighted method: given x^k at an iteration, it generates x^{k+1} as the unique solution of the reweighted l_1 norm minimization problem: $\min_x \ \lambda\|W^k x\|_1 + \|Ax - b\|_2^2$, where $W^k = \text{Diag}\{(|x_i^k| + \varepsilon)^{p-1} : i = 1, \ldots, n\}$. Such a reweighted l_1 norm minimization algorithm for sparse signal recovery was originally introduced by Candés, Wakin and Boyd [15]. In 2009, Foucart and Lai [6] proved that under the assumption of Restricted Isometry Property (RIP for short) condition, the reweighted l_1 norm minimization algorithm with weight $w_i^k = (|x_i^k| + \varepsilon)^{p-1}$ for any index $i \in \{1, \ldots, n\}$ can exactly recover the sparse signal, where $p \in (0, 1)$ is a given parameter. Recently, Chen and Zhou [16] gave an unconstrained iteratively reweighted l_1 minimization algorithm to solve (1.5), and further proved that under RIP/NSP(Null Space Property) type conditions the accumulation points of the sequence generated by the reweighted l_1 norm minimization algorithm can converge to the stationary point of (1.5). Moreover, Chen and Zhou [17] also proposed an effective globally convergent smoothing nonlinear conjugate gradient method for solving nonconvex minimization problems, which can guarantee that any accumulation point of a sequence generated by this method is a Clarke stationary point of the problem concerned.

In the context of the matrix case, Mohan and Fazel [18] proposed an iteratively reweighted least square (IRLS for short) algorithm (see, e.g., [18–21]) for minimizing the smooth Schatten-p function $f_p(X) = \text{Tr}(X^T X + \gamma I)^{p/2}$, i.e.,

$$\min_X \ f_p(X) \quad \text{s.t.}\, \mathcal{A}(X) = b \text{ where } 0 < p < 1.$$

Recently, Lai et al. [21] considered the unconstrained minimization problem:

$$\min_X \ \text{Tr}(X^T X + \varepsilon^2 I)^{p/2} + \frac{1}{2\lambda}\|\mathcal{A}(X) - b\|_2^2 \quad \text{where } 0 < p < 1.$$

Given W^k at an iteration, the IRLS algorithm [21] generates X^{k+1} by

$$X^{k+1} := \arg\min_X \ \text{Tr}(W_p^k X^T X) + \frac{1}{\lambda}\|\mathcal{A}(X) - b\|_2^2,$$

where $W_p^k = p(X^{k^T} X^k + \varepsilon^{k^2} I)^{p/2-1}$. Both in [18,21], the iteration of regular parameters γ and ε in the algorithm take the same rule as the one in [22], i.e., $\gamma^k = \min\{\gamma^{k-1}, \gamma_s \sigma_{r+1}(X^k)\}$ and $\varepsilon^k = \min\{\varepsilon^{k-1}, \gamma_s \sigma_{r+1}(X^k)\}$ with $\gamma_s < 1$ and r being a guesstimate of the rank.

In this paper, we first introduce an unconstrained smooth nonconvex approximation of problem (1.1) by the following L_2–M_p problem:

$$\min_X \ \lambda\|X\|_{p,\varepsilon} + \|\mathcal{A}(X) - b\|_2^2, \tag{1.6}$$

where λ is a parameter which is sufficiently small, and $\|X\|_{p,\varepsilon} := \sum_{i=1}^n (\sigma_i(X) + \varepsilon)^p$ with $\varepsilon > 0$ and $p \in (0, 1)$. For any fixed scalar $\varepsilon > 0$ and $p \in (0, 1)$, $\|\cdot\|_{p,\varepsilon}$ is subdifferentiable; and it satisfies that $\|X\|_{p,\varepsilon} \to \text{rank}(X)$ as $\varepsilon \to 0$ and $p \to 0$. We propose a reweighted nuclear norm minimization algorithm for solving problem (1.6), i.e.,

$$X^{k+1} := \arg\min_X \lambda p \sum_{i=1}^n w_i^k \sigma_i(X) + \|\mathcal{A}(X) - b\|_2^2, \tag{1.7}$$

where $w_i^k = (\sigma_i(X^k) + \varepsilon)^{p-1}$ for all $i \in \{1, 2, \ldots, n\}$ and $\sigma_i(X^k)$ is the i-th singular value of X^k. We call the weighted form of $\sum_{i=1}^n w_i^k \sigma_i(X)$ as the weighted nuclear norm. It is not a norm strictly speaking, but it is convex. Thus, the iteration (1.7) is called a reweighted nuclear norm minimization algorithm. For the implementation of the iteration (1.7), how to solve