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On graded meshes for weakly singular Volterra integral equations with oscillatory trigonometric kernels☆

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ABSTRACT

We compare the collocation methods on graded meshes with that on uniform meshes for the solution of the weakly singular Volterra integral equation of the second kind with oscillatory trigonometric kernels. Due to, in general, unbounded derivatives at the left endpoint of the interval of integration, we should approximate the solution of the integral equation by collocation methods on graded meshes. However, we show that this non-smooth behavior of the solution has little effect on the approximate solution when the kernels of integral equation are highly oscillatory. Numerical examples are given to confirm the proposed results.

1. Introduction

The Volterra integral equation of the second kind

$$y(x) + \int_0^x \frac{K(x,t)}{(x-t)^{\alpha}} y(t)dx = f(x), \quad x \in [0,T],\ \alpha < 1, \tag{1.1}$$

where $K(x,t)$ is a continuous function and $f(x)$ is a given function, arises in the study of various problems of mathematical physics and engineering [1–3]. Particularly, when $K(x,x) \neq 0$ and $0 < \alpha < 1$, (1.1) is called weakly singular.

The theoretical aspects of the solutions of the general weakly singular Volterra integral equation (1.1) had been widely studied, and the existence and uniqueness of a solution had been established in certain cases, for details see [4–6]. However, in most of the cases, the integral equation cannot be done analytically and one has to resort to numerical methods. For solving such an integral equation, Galerkin methods and collocation methods and its variants are very popular [7,8,2,9,10]. The principal idea of collocation methods is achieved by approximating the solution of integral equation $y(x)$ by a piecewise polynomial $p(x)$. For details we refer to [7,11,2,12].

In (1.1), when $\alpha < 0$, the global convergence order of the polynomial collocation method is $O(h^m)$ for the uniform meshes [7], where h is the diameter of the mesh, $m-1$ is the degree of the approximating polynomials. While $0 < \alpha < 1$ the kernel of Eq. (1.1) is non-smoothing, and the convergence order of the polynomial collocation method is only $O(h^{1-\alpha})$ for uniform meshes, independent of the degree of the polynomials [7,8,2,13]. To cope with this problem, one has to use

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Table 1
Absolute error $\max_{x\in I_h}|\hat{y}(x)-y(x)|$ for $y(x)+\int_0^x \frac{1}{\sqrt{x-t}}y(t)dt=1$.

	$G_{10}^{L,0}$	$U_{10}^{L,0}$	$G_{10}^{L,1}$	$U_{10}^{L,1}$
$\max_{x\in I_h}\lvert\hat{y}(x)-y(x)\rvert$	0.141×10^{-1}	0.266×10^{-1}	0.202×10^{-2}	0.308×10^{-1}

Table 2
Absolute error $\max_{x\in I_h}|\hat{y}(x)-y(x)|$ for $y(x)+\int_0^x \frac{1}{\sqrt{x-t}}y(t)dt=1$.

	$G_{10}^{L,2}$	$U_{10}^{L,2}$	$G_{10}^{L,3}$	$U_{10}^{L,3}$
$\max_{x\in I_h}\lvert\hat{y}(x)-y(x)\rvert$	0.152×10^{-3}	0.104×10^{-2}	0.976×10^{-5}	0.509×10^{-3}

Table 3
Approximations for $y(x)+\int_0^x \frac{e^{i\omega(x-t)}}{\sqrt{x-t}}y(t)dt=\exp(x)$.

$\max_{x\in I_h}\lvert\hat{y}(x)-y(x)\rvert$	$\omega=1$	$\omega=100$	$\omega=10^4$	$\omega=10^8$
$G_{10}^{L,0}$	0.414×10^{-1}	0.144×10^{-1}	0.246×10^{-3}	0.270×10^{-7}
$U_{10}^{L,0}$	0.249×10^{-1}	0.155×10^{-1}	0.208×10^{-3}	0.217×10^{-7}
$G_{10}^{L,1}$	0.618×10^{-2}	0.741×10^{-2}	0.229×10^{-3}	0.307×10^{-7}
$U_{10}^{L,1}$	0.302×10^{-1}	0.150×10^{-1}	0.180×10^{-3}	0.184×10^{-7}
$G_{10}^{L,2}$	0.112×10^{-3}	0.562×10^{-2}	0.292×10^{-3}	0.143×10^{-5}
$U_{10}^{L,2}$	0.105×10^{-2}	0.138×10^{-1}	0.180×10^{-3}	0.184×10^{-7}

Table 4
Approximations for $y(x)+\int_0^x \frac{e^{i\omega(x-t)}}{(x-t)^{0.1}}y(t)dt=\exp(x)$.

$\max_{x\in I_h}\lvert\hat{y}(x)-y(x)\rvert$	$\omega=1$	$\omega=100$	$\omega=10^4$	$\omega=10^8$
$G_{10}^{L,0}$	0.111	0.953×10^{-2}	0.134×10^{-3}	0.821×10^{-8}
$U_{10}^{L,0}$	0.281×10^{-1}	0.874×10^{-2}	0.102×10^{-3}	0.859×10^{-8}
$G_{10}^{L,1}$	0.269×10^{-2}	0.640×10^{-2}	0.694×10^{-4}	0.694×10^{-8}
$U_{10}^{L,1}$	0.999×10^{-3}	0.650×10^{-2}	0.694×10^{-4}	0.694×10^{-8}
$G_{10}^{L,2}$	0.105×10^{-4}	0.566×10^{-2}	0.680×10^{-4}	0.677×10^{-8}
$U_{10}^{L,2}$	0.315×10^{-4}	0.680×10^{-2}	0.694×10^{-4}	0.694×10^{-8}

suitable graded meshes or use non-polynomial approximating functions reflecting the behavior of the exact solution near $t=0$. For a detailed analysis of the collocation method on graded meshes see [8]. However, for a highly oscillatory integral equation, the convergence rate is not only determined by the meshes but also by the frequency ω. For $\omega\gg1$, numerical experiments show that collocation on uniform meshes can achieve the same precision with less computational cost as that on graded meshes, see Tables 3–6.

On the other hand, the efficient evaluation of the solution of (1.1) is based on the efficient computation of the integrals occurring in the collocation equation, which usually cannot be found analytically but have to be approximated by suitable numerical quadrature formulas. Specially, when $K(x,t)$ is a highly oscillatory function, where standard quadrature methods are exceedingly difficult and the cost steeply increases with the frequency, such as $K(x,t)=e^{i\omega cos(x-t)}$, $i=\sqrt{-1}$, $\omega\gg1$.

In the last few years many efficient methods have been devised for the evaluation of the oscillatory integral $\int_a^b f(x)e^{i\omega g(x)}dx$, such as the asymptotic method [14], Filon-type method [15], Levin's collocation method [16], modified Clenshaw–Curtis method, Clenshaw–Curtis–Filon-type method [17], and generalized quadrature rule [18]. Recently, H. Kang and S. Xiang presented some quadrature methods for a class of highly oscillatory integrals with singularities at the two endpoints of the interval, for details see [19]. In many situations the accuracy of the Filon-type method is significantly higher than that of the asymptotic method, even though it is of the same order. But the Filon-type method requires that the moments $\int_a^b x^k e^{i\omega g(x)}dx$ are easily computable, which is not necessarily the case. To work around this weakness, Xiang [20] derived an efficient Filon-type method, an approach without computing the moments. Also, Olver presented a moment-free method, for details see [21,22].