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A two-level finite volume method for the unsteady Navier–Stokes equations based on two local Gauss integrations[☆]

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ABSTRACT

In this paper we consider a two-level finite volume method for the two-dimensional unsteady Navier–Stokes equations by using two local Gauss integrations. This new stabilized finite volume method is based on the linear mixed finite element spaces. Some new a priori bounds for the approximate solution are derived. Moreover, a two-level stabilized finite volume method involves solving one small Navier–Stokes problem on a coarse mesh with mesh size H, a large general Stokes problem on the fine mesh with mesh size $h \ll H$. The optimal error estimates of the H^1-norm for velocity approximation and the L^2-norm for pressure approximation are established. If we choose $h = \mathcal{O}(H^2)$, the two-level method gives the same order of approximation as the one-level stabilized finite volume method. However, our method can save a large amount of computational time.

1. Introduction

Finite volume method (FVM) as one of the important numerical discretization techniques has been widely employed to solve the fluid dynamics problems [1]. It is developed as an attempt to use finite element idea in the finite difference setting. The basic idea is to approximate the discrete fluxes of a partial differential equation using a finite element procedure based on volumes or control volumes, so FVM is also called box scheme, generalized difference method and so on (see [2,1]). FVM has many advantages that belong to finite difference or finite element methods, such as, it is easy to set up and implement, conserve mass locally and FVM also can treat the complicated geometry and general boundary conditions flexibility. However, the analysis of FVM lags far behind than that of finite element and finite difference methods, we can refer to [3–8] and the references therein for more recent development about the finite volume method.

In this work, we consider the unsteady incompressible Navier–Stokes equations

$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, & \text{in } \Omega \times (0, T], \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, T], \\ u = 0 & \text{on } \partial\Omega \times (0, T], \\ u = u_0 & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.1}$$

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where Ω be a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$, $u = (u_1(x,t), u_2(x,t))^T$ the velocity, $p = p(x,t)$ the pressure, $f = f(x,t) \in L^2(\Omega)^2$ the prescribed body force, $\nu > 0$ the viscosity, u_0 the initial velocity, $T > 0$ a finite time, and $u_t = \partial u/\partial t$.

The development of an efficient finite element method for the Navier–Stokes equations is an important but challenging problem in incompressible flow simulations. The importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying the so-called inf–sup condition is widely known. Although some stable mixed finite element pairs have been studied over the years [9,10], while the P_1–P_1 pair not satisfying the inf–sup condition may also work well. The P_1–P_1 pair is computationally convenient in a parallel processing and multigrid context because this pair holds the identical distribution for both velocity and pressure. Moreover, the P_1–P_1 pair is of practical importance in scientific computation with the lowest computational cost. Therefore, much attention has been attracted by the P_1–P_1 pair for simulating the incompressible flow, we can refer to [11–14] and the references therein.

In order to use the unstable mixed finite element pairs, various stabilized techniques have been proposed and studied. For example, the polynomial pressure projection method [11,14], the stream upwind Petrov–Galerkin (SUPG) method [15], the Douglas–Wang method [16], the macro-element method [17] and so on. Most of these stabilized methods necessarily introduce the stabilization parameters either explicitly or implicitly. In addition, some of these techniques are conditionally stable or are of suboptimal accuracy. Therefore, the development of the efficient stabilized methods free from stabilization parameters has become increasingly important.

Recently, a family of stabilized finite element methods for the Stokes problem have been established in [18] by using the polynomial pressure projections, authors not only presented the stabilized discrete formulation for the Stokes equations but also obtained the optimal error estimates. Compared with the other stabilized methods which mentioned above, this new stabilized method has the following features: parameter-free, avoiding higher-order derivatives or edge-based data structures and unconditionally stable. Based on the ideas of [18,11], by using the difference between two local Gauss integrations as the component for pressure, Li et al. developed another new kind of stabilized method for the incompressible flow problem based on the linear function spaces [19,20,13,4], and their method can be applied to the existing codes with a little additional effort.

On the other hand, the two-level method is an efficient numerical scheme for the partial differential equations based on two spaces with different mesh sizes. This kind of discretization technique for linear and nonlinear elliptic partial differential equations was first introduced by Xu in [21,22]. After then, two-level scheme has been studied by many researchers, for example, Dawson et al. studied the nonlinear parabolic equations by using the finite element or finite difference methods in [23,24], respectively. Layton and Leferink [25] for Navier–Stokes equations, Bi and Ginting [26] expanded the two-level scheme combined with finite volume method for linear and nonlinear elliptic problems. Recently, we studied the stability and convergence of two-level finite volume methods for the nonlinear parabolic in semidiscrete and fully discrete formulations in [6–8], respectively.

In this paper, we devote ourselves to the research of the two-level finite volume method for the unsteady Navier–Stokes problem. By introducing a pressure polynomial projection operator from linear space to the constant space, some new a priori bounds of numerical solution are established. Another important novel ingredient of this work is the convergence analysis of the approximate solution in two-level scheme. For the one-level finite volume scheme, which involves solving one large nonlinear problem on a fine mesh with mesh size h, we have the following error estimate for the numerical solution (u_h, p_h):

$$\nu\|\nabla(u-u_h)\|_0 + \|p-p_h\|_0 \le Ch, \tag{1.2}$$

where $C > 0$ is a generic constant, it may stand for different values at its different occurrences. Furthermore, we present the two-level finite volume solution $(u_h^\upsilon, p_h^\upsilon)$ in the following error estimate:

$$\sigma^{\frac{1}{2}}(t)(\nu\|\nabla(u-u_h^\upsilon)\|_0 + \|p-p_h^\upsilon\|_0) \le C(h+H^2). \tag{1.3}$$

Hence, if we choose H such that $h = \mathcal{O}(H^2)$ for the two-level stabilized finite volume formulation, then the method we studied is of the same convergence order as that of the one-level finite volume method. However, our method is simpler than the one-level FVM.

2. Function setting for the Navier–Stokes equations

For the mathematical setting of problem (1.1), we set

$$X = H_0^1(\Omega)^2, \qquad Y = L^2(\Omega)^2, \qquad D(A) = H^2(\Omega)^2 \cap X,$$
$$M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q\,dx = 0 \right\}.$$

The spaces $L^2(\Omega)^m$ $(m = 1, 2, 4)$ are endowed with the standard L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. The spaces $H_0^1(\Omega)$ and $H_0^1(\Omega)^2$ are equipped with the scalar product $(\nabla u, \nabla v)$ and norm $\|u\|_1^2 = (\nabla u, \nabla u)$, $\forall\, u, v \in H_0^1(\Omega)$ or $H_0^1(\Omega)^2$.

We introduce the Laplace operator $Au = -\Delta u$, $\forall\, u \in D(A)$. For the initial data u_0 and the body force f, we recall the following assumption.