Journal of Computational and Applied Mathematics 263 (2014) 445–465

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Rational Hausdorff divisors: A new approach to the approximate parametrization of curves

Sonia L. Rueda [a], Juana Sendra [b,c], J. Rafael Sendra [d,*]

[a] Dpto. de Matemática Aplicada, E.T.S. Arquitectura, Universidad Politécnica de Madrid, E-28040 Madrid, Spain

[b] Dpto. Matemática Aplicada I.T. Telecomunicación. UPM, Spain

[c] Research Center on Software Technologies and Multimedia Systems for Sustainability (CITSEM), Spain

[d] Dpto. de Fisica y Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain

ARTICLE INFO

Article history:
Received 18 April 2013
Received in revised form 4 November 2013

Keywords:
Hausdorff distance
Rational Hausdorff divisor
Hausdorff curve
Rational curve
Approximate parametrization problem

ABSTRACT

In this paper, we introduce the notion of rational Hausdorff divisor, we analyze the dimension and irreducibility of its associated linear system of curves, and we prove that all irreducible real curves belonging to the linear system are rational and are at finite Hausdorff distance among them. As a consequence, we provide a projective linear subspace where all (irreducible) elements are solutions of the approximate parametrization problem for a given algebraic plane curve. Furthermore, we identify the linear system with a plane curve that is shown to be rational and we develop algorithms to parametrize it analyzing its fields of parametrization. Therefore, we present a generic answer to the approximate parametrization problem. In addition, we introduce the notion of Hausdorff curve, and we prove that every irreducible Hausdorff curve can always be parametrized with a generic rational parametrization having coefficients depending on as many parameters as the degree of the input curve.

1. Introduction

When applying computational mathematics in practical applications, even though one may be dealing with a problem that can be solved algorithmically, and even though one has good algorithms to approach the solution, it can happen, and often it is the case, that the problem has to be reformulated and analyzed from a different computational point of view. This is the case of the development of approximate algorithms. This paper frames in the research area of approximate algebraic geometry and commutative algebra and, more precisely, on the problem of the approximate parametrization.

1.1. Approximate algebraic geometry and commutative algebra

We start with a subsection devoted to introduce informally the idea of approximate algorithm and to comment on some achievements in approximate algebraic geometry and commutative algebra.

Let $\mathcal{E}$ be a mathematical entity appearing in the resolution of a practical problem (e.g. $\mathcal{E}$ is a real polynomial) that is known, because of the nature of the treated applied problem, to satisfy certain property $\mathcal{P}$ (e.g. being reducible over $\mathbb{Q}$) that implies the existence of certain associated objects $\mathcal{E}_1, \ldots, \mathcal{E}_n$ (e.g. the irreducible factors over $\mathbb{Q}$ of the polynomial), and let the goal of the problem be to compute $\mathcal{E}_1, \ldots, \mathcal{E}_n$. However, often in practical applications, we receive a perturbation

* Corresponding author.
E-mail addresses: sonialuisa.rueda@upm.es (S.L. Rueda), jsendra@euitt.upm.es (J. Sendra), rafael.sendra@uah.es (J. Rafael Sendra).

0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.12.052

$\mathcal{E}'$ of $\mathcal{E}$ instead of $\mathcal{E}$, where the property $\mathcal{P}$ does not hold anymore neither the associated objects $\mathcal{E}_i$ exist; for instance, a perturbation of a $\mathbb{Q}$-reducible polynomial will be, in general, $\mathbb{Q}$-irreducible and, therefore, the application of the existing polynomial factorization algorithms will just not solve our problem. One may try to recover the original unperturbed entity $\mathcal{E}$. Since, this is essentially impossible, a more realistic version of the problem is to determine a new object $\mathcal{E}''$ near $\mathcal{E}'$, satisfying $\mathcal{P}$, as well as computing the associated objects $\mathcal{E}_i''$ to $\mathcal{E}''$. An algorithm solving a problem of the above type is called an approximate algorithm; a solution of the illustrating example on polynomial factorization is given e.g. in [1].

One may distinguish two main phases when dealing with this type of problems. On one hand, the development of a theoretical reasoning that yields an algorithm and, on the other, providing an analysis of the distance between input and output in terms of a given tolerance. The distance used depends on the particular treated problem. For instance, in the factorization problem, if the input is $f \in \mathbb{C}[x_1, \dots, x_n]$, and the output is $g \in \mathbb{C}[x_1, \dots, x_n]$, one requires that $\|f - g\|_\infty$ is smaller than the tolerance. Algebraic varieties (that is, sets of points whose coordinates are zeros of some polynomials), as for instance algebraic curves and algebraic surfaces, are the main objects in algebraic geometry. Therefore, in approximate algebraic geometry (which is our case), in general, since one usually deals with sets, the Hausdorff distance is used. Indeed, the Hausdorff distance has proven to be an appropriate tool for measuring the resemblance between two geometric objects, becoming in consequence a widely used tool in fields as computer aided design, pattern matching and pattern recognition (see e.g. [2,3]); at the end of Section 1.2 we recall the definition of Hausdorff distance.

One can find in the literature papers treating this type of problems (say related to algebraic geometry and commutative algebra) with the same, or similar, strategies; see [4] for a general overview on approximate commutative algebra. Approximate algorithms for computing polynomial gcds can be found in [5–7], the polynomial factorization problem is addressed in [8,9,1,10]. For algebraic varieties there also exist approximate solutions: see [11–13] for the implicitization problem, see [14–16] for the study and analysis of singularities, and for the parametrization problem see [17–26]; approximate parametrization is the central topic of this paper and it will be commented in detail in the next subsections of this introduction.

1.2. Approximate parametrization problem

The approximate parametrization problem can be stated as follows (we state it for plane curves, but it can be similarly stated for space curves, see [25], and for surfaces, see [24]).

> Approximate parametrization problem for plane curves: Given the implicit equation of a non-rational real plane curve $\mathcal{C}$ and a tolerance $\epsilon > 0$, decide whether there exists a rational real plane curve $\overline{\mathcal{C}}$ at finite small distance (i.e. small related to the tolerance ϵ) to the input curve $\mathcal{C}$ and, in the affirmative case, compute a real rational parametrization of $\overline{\mathcal{C}}$.

The problem, as stated above, requires a global answer, that is, an algebraic curve $\overline{\mathcal{C}}$ to play the role of $\mathcal{C}$. However, not always such an answer exists. For instance, if one is interested in getting $\overline{\mathcal{C}}$ with the same topological graph as $\mathcal{C}$, the expected answer is that there will be no global solution; note that the genus of $\overline{\mathcal{C}}$ and $\mathcal{C}$ are different and the genus measures the difference between the maximum number of singularities and the actual one, counted properly, and this clearly affects the graph. So, the problem is often reformulated such that the solution is given in terms of piecewise rational curves; see for instance [18–21,26]. We, in our research (see [22–25]), are interested in the global answer, and hence, in the problem as it is stated above.

In our situation, the input curve $\mathcal{C}$ is supposed to have suffered a perturbation, in the sense that the coefficients of its defining polynomials have been perturbed. To treat the problem, even though the coefficients of our input polynomial have suffered a perturbation, and hence they are not the correct expected ones, we consider them as exact coefficients of our input and we work symbolically with them. For instance, let us consider the curve $\mathcal{C}$ defined by the implicit equation

$$x^4 + 2\,y^4 + \frac{1001}{1000}\,x^3 + 3\,x^2y - y^2x - 3\,y^3 + \frac{1}{100000}\,y^2 - \frac{1}{1000}\,x - \frac{1}{1000}\,y - \frac{1}{1000} = 0$$

and the curve $\overline{\mathcal{C}}$ defined by the implicit equation

$$x^4 + 2\,y^4 + \frac{1001}{1000}\,x^3 + 3\,x^2y - y^2x - 3\,y^3 = 0.$$

The curve $\mathcal{C}$ cannot be parametrized by means of rational functions, indeed its genus is 3, while $\overline{\mathcal{C}}$ can be parametrized e.g. by

$$\left(\frac{3000\,t^3 + 1000\,t^2 - 3000\,t - 1001}{1000(2\,t^4 + 1)}, \frac{t\left(3000\,t^3 + 1000\,t^2 - 3000\,t - 1001\right)}{1000(2\,t^4 + 1)}\right),$$

and one can see in Fig. 1 that both curves are close to each other.

But, above, what does it mean that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are close? As mentioned before, we require that the Hausdorff distance between $\mathcal{C}, \overline{\mathcal{C}}$ is small related to a given tolerance ϵ. A main difficulty when working with the Hausdorff distance is that, if not both sets are bounded, the distance between them can be infinity. Most of the papers deal with bounded real algebraic