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Quasi-optimal rates of convergence for the Generalized Finite Element Method in polygonal domains

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ABSTRACT

We consider a mixed-boundary-value/interface problem for the elliptic operator $P = -\sum_{ij} \partial_i(a_{ij}\partial_j u) = f$ on a polygonal domain $\Omega \subset \mathbb{R}^2$ with straight sides. We endowed the boundary of Ω partially with Dirichlet boundary conditions $u = 0$ on $\partial_D\Omega$, and partially with Neumann boundary conditions $\sum_{ij} \nu_i a_{ij}\partial_j u = 0$ on $\partial_N\Omega$. The coefficients a_{ij} are piecewise smooth with jump discontinuities across the interface Γ, which is allowed to have singularities and cross the boundary of Ω. In particular, we consider "triple-junctions" and even "multiple junctions". Our main result is to construct a sequence of Generalized Finite Element spaces S_n that yield "h^m-quasi-optimal rates of convergence", $m \geq 1$, for the Galerkin approximations $u_n \in S_n$ of the solution u. More precisely, we prove that $\|u - u_n\| \leq C \dim(S_n)^{-m/2}\|f\|_{H^{m-1}(\Omega)}$, where C depends on the data for the problem, but not on f, u, or n and $\dim(S_n) \to \infty$. Our construction is quite general and depends on a choice of a good sequence of approximation spaces S'_n on a certain subdomain W that is at some distance to the vertices. In case the spaces S'_n are Generalized Finite Element spaces, then the resulting spaces S_n are also Generalized Finite Element spaces.

Introduction

The purpose of this work is to present a general construction of finite-dimensional approximation spaces S_n that yields quasi-optimal rates of convergence for the Galerkin approximation of the solution to an elliptic equation in a polygonal domain, when mixed Dirichlet–Neumann conditions are given at the boundary. The coefficients of the equation are piecewise smooth, but may have *jump discontinuities* across the union of a finite number of closed polygonal lines, which we call the *interface*. The interface may intersect the boundary of the polygonal domain.

The construction of the Galerkin spaces S_n employs a sequence of local spaces S'_n with good approximation properties given on a subset of Ω at a positive distance from the singular points of the domain and the interface. Once S'_n are chosen, grading towards the vertices and suitable partitions of unity are employed to define the Galerkin spaces on the whole domain. Therefore, the construction of the spaces S_n falls into the category of Generalized Finite Element Methods (GFEM) and do not require any particular meshing of the domain in advance.

We next describe the problem and the geometric set-up more precisely. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with straight sides (we will call it a *straight polygonal domain*). We assume that $\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}_k$, where Ω_k are disjoint straight polygonal domains. The set $\Gamma := \partial\Omega \setminus \cup_{j=1}^{K} \partial\Omega_k$, that is, the part of the boundary of some Ω_k that is not contained in the boundary of

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Ω, will be called *the interface*. We then consider the following boundary value problem:

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega \\ \nu \cdot A \cdot \nabla u = 0 & \text{on } \partial_N \Omega \\ u = 0 & \text{on } \partial_D \Omega, \end{cases} \tag{0.1}$$

where $\partial_N\Omega := \partial\Omega \backslash \partial_D\Omega$ is a decomposition $\partial\Omega$ into two disjoint sets, with $\partial_D\Omega$ a finite union of closed straight segments, and ν is the unit outer normal to Ω, defined everywhere except at the vertices. We assume that the differential operator $P := -\operatorname{div} A\nabla = \sum_{ij} \partial_i a_{ij} \partial_j$ is *uniformly strongly elliptic* and that its coefficients a_{ij} are piecewise smooth, but may jump across the interface Γ. For this reason, we will refer to Problem (0.1) as a *mixed boundary value/interface problem on* Ω.

Mixed boundary value/interface problems often appear in engineering and physics. It is well-known that if Ω is convex, $f \in L^2(\Omega)$, and the coefficient matrix $A = [a_{ij}]$ is smooth on $\overline{\Omega}$ (so there is no interface), then the solution u of (0.1) is in $H^2(\Omega)$, and we can get quasi-optimal rates of convergence for the standard Finite Element Method (FEM) with piecewise linear polynomials and quasi-uniform meshes. When Ω is not convex and the boundary has singularities or the matrix A is discontinuous, on the other hand, then u does not belong to $H^2(\Omega)$ and we may obtain decreased rates of convergence of the Finite Element approximations of u on quasi-uniform meshes. Here and throughout the paper, we denote by $H^m(\Omega)$, $m \in \mathbb{Z}_+$, the standard L^2-based Sobolev spaces.

Finding efficient methods to treat mixed boundary value problem on straight polygons using Generalized Finite Element Method is part of the general problem of numerically treating singularities. If the coefficients a_{ij} are smooth on each subdomain Ω_j, then singularities arise only at the vertices of the domain Ω, at the points where the boundary conditions change, and at the singular points of the interface or where the interface touches the boundary. Additional singularities will arise if some of the coefficients a_{ij} or the data f are singular at some other points. In this paper, however, we shall assume that our coefficients are piecewise smooth and that data is regular, i.e., $f \in H^{m-1}(\Omega)$, $m \geq 1$.

The structure of corner singularities in two dimensional space is well known by the works [1,2] and many others. (See, for instance [2–7], for more information about singularities that are especially relevant to this paper). Singularities in the solution in the neighborhood of a corner are determined by the spectrum of the resulting pencil of elliptic operators obtain through the Mellin Transform [2,8].

The FEMs and GFEMs are examples of Galerkin-based numerical methods, a concept we briefly recall. It is based on the weak formulation of problem (0.1), which is discussed in Section 1. Suppose we are given a sequence of finite-dimensional spaces $S_n \subset H^1(\Omega)$ such that all the functions $\psi \in S_n$ satisfy the essential (i.e., Dirichlet) boundary conditions of Eq. (0.1) on $\partial_D\Omega$. For the simplicity of the presentation, we shall assume that $\partial_D\Omega$ is not empty. That is, we do not consider the *pure Neumann problem* explicitly. To consider also the pure Neumann problem, all that one needs to do in practice is to restrict to functions $v \in S_n$ with zero mean. We define, as usual, the Galerkin approximation $u_n \in S_n$ of the variational solution u of Problem (0.1), to be the exact solution of the projected problem:

$$B(u_n, v_n) := \sum_{ij} \int_\Omega a_{ij} \partial_i u_n \partial_j v_n = (f, v_n), \quad \text{for all } v_n \in S_n \subset H^1_D(\Omega), \tag{0.2}$$

where $H^1_D(\Omega) := \{f \in H^1(\Omega), f = 0 \text{ on } \partial_D\Omega\}$ and the bilinear form $B(u, v)$ is given in (1.5). We want a h^m-*quasi-optimal rate of convergence*, that is, we want to have the following error estimate for all n

$$\|u - u_n\|_{H^1(\Omega)} \leq C \, \dim(S_n)^{-m/2} \|f\|_{H^{m-1}},$$

where C is independent of f and n. Up to the value of C this is the ideal rate that can be obtained if $u \in H^{m+1}(\Omega)$ and quasi-uniform meshes are used in the Finite Element Method. In this case, if h is the typical size of an element, then $\dim(S_n) \sim h^{-2}$ hence the name. However, in general, $u \notin H^{m+1}(\Omega)$. If Ω is concave, even $u \notin H^2(\Omega)$ in general for the standard Poisson problem. In fact, as mentioned, singularities may lower the rate of convergence of the Finite Element solutions of the discrete problem when using quasi-uniform meshes.

Our approach to the optimal rate of convergence is based on the weighted Sobolev spaces for mixed boundary value and interface problems on polygonal domains, obtained by two of the authors among others [6,9–11], and a grading towards each singular point. The weight is here the distance to the singular set. Our main result is to construct a sequence S_n of the Generalized Finite Element spaces that yields quasi-optimal rates of convergence. We are not assuming $u \in H^{m+1}(\Omega)$ and we can relax the condition $f \in H^{m-1}(\Omega)$ to $f \in \hat{H}^{m-1}(\Omega) := \sum H^{m-1}(\Omega_j)$ if an interface is present.

As we mentioned above, we use some auxiliary, "good approximation spaces" S'_n, defined on an auxiliary, but fixed domain W away from the vertices. Together with grading towards the singular point and partitions of unity, the spaces S'_n lead to the construction of the Galerkin spaces S_n that then yield our desired h^m-quasi-optimal rates of convergence. Many choices for the spaces S'_n exist [6,12–14], and their definition is, for the most, part very well known, so we do not recall them here.

We notice, however, that if the sequence S'_n is a sequence of GFEM spaces, then S_n will also be a sequence of GFEM spaces. However, if S'_n consists of FEM spaces, then S_n will not consist of FEM spaces, in general. In fact, we mention two examples that satisfy the required approximation condition (see (2.5)). One example is that of FEM spaces consisting of piecewise linear elements on a sequence of appropriately graded meshes, where the grading is determined by the strength of the