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A unified approach to pricing and risk management of equity and credit risk

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ABSTRACT

We propose a unified framework for equity and credit risk modeling, where the default time is a doubly stochastic random time with intensity driven by an underlying affine factor process. This approach allows for flexible interactions between the defaultable stock price, its stochastic volatility and the default intensity, while maintaining full analytical tractability. We characterize all risk-neutral measures which preserve the affine structure of the model and show that risk management as well as pricing problems can be dealt with efficiently by shifting to suitable survival measures. As an example, we consider a jump-to-default extension of the Heston stochastic volatility model.

1. Introduction

The last few years have witnessed an increasing popularity of hybrid equity/credit risk models, as documented by the recent papers [1–8]. One of the most appealing features of such models is their capability to link the stochastic behavior of the stock price (and of its volatility) with the randomness of the default event and, hence, with the level of credit spreads. The relation between equity and credit risk is supported by strong empirical evidence (we refer the reader to [3,8] for good overviews of the related literature) and several studies document significant relationships between stock volatility and credit spreads of corporate bonds and Credit Default Swaps [9,10].

In this paper, we propose a general framework for the joint modeling of equity and credit risk which allows for a flexible dependence between stock price, stochastic volatility, default intensity and interest rate. The proposed framework is fully analytically tractable, since it relies on the powerful technology of affine processes (see e.g. [11,12] for financial applications of affine processes), and nests several stochastic volatility models proposed in the literature, thereby extending their scope to a defaultable setting. Affine models have been successfully employed in credit risk models, as documented by the papers [7,13,14]. A distinguishing feature of our approach is that, unlike the models proposed in [1,3–5,7,8], we jointly consider both physical and risk-neutral probability measures, ensuring that the analytical tractability is preserved under a change of measure, while at the same time avoiding unnecessarily restrictive specifications of the risk premia. This aspect is of particular importance in credit risk modeling, where one is typically faced with the two problems of computing survival probabilities or related risk measures and of computing arbitrage-free prices of credit derivatives. In this paper, we provide a complete characterization of the set of risk-neutral measures which preserve the affine structure of the model, thus enabling us to efficiently compute several quantities which are of interest in view of both risk management and pricing applications.

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The paper is structured as follows. Section 2 introduces the modeling framework, while Section 3 gives a characterization of the family of risk-neutral measures which preserve the affine structure of the model. In Sections 4–5, we show how most quantities of interest for risk management and pricing applications, respectively, can be efficiently computed under suitable (risk-neutral) survival measures (we refer the reader to Sect. 2.5 of [15] for more detailed proofs of the results of Sections 4–5). Section 6 illustrates the main features of the proposed approach within a simple example, which corresponds to a defaultable extension of the Heston [16] model. Finally, Section 7 concludes.

2. The modeling framework

This section presents the mathematical structure of the modeling framework. Let $(\Omega, \mathcal{G}, P)$ be a reference probability space, with P denoting the physical/statistical probability measure (we want to emphasize that our framework will be entirely formulated with respect to the physical measure P). Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{0\le t\le T}$ an $\mathbb{R}^d$-valued Brownian motion on $(\Omega, \mathcal{G}, P)$, with $d \ge 2$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ its P-augmented natural filtration.

We focus our attention on a single defaultable firm, whose *default time* $\tau : \Omega \to [0, T]\cup\{+\infty\}$ is supposed to be a $(P, \mathbb{F})$-*doubly stochastic* random time, in the sense of Def. 9.11 of [17]. This means that there exists a strictly positive $\mathbb{F}$-adapted process $\lambda^P = (\lambda^P_t)_{0\le t\le T}$ such that

$$P(\tau > t|\mathcal{F}_T) = P(\tau > t|\mathcal{F}_t) = \exp\left(-\int_0^t \lambda^P_u du\right), \quad \text{for all } t \in [0, T].$$

In order to emphasize the role of the reference measure P, we call the process λ^P the *P-intensity* of τ. Let the filtration $\mathbb{G} = (\mathcal{G}_t)_{0\le t\le T}$ be the *progressive enlargement*[1] of $\mathbb{F}$ with respect to τ, i.e., $\mathcal{G}_t := \bigcap_{s>t}\left\{\mathcal{F}_s \vee \sigma(\tau\wedge s)\right\}$, for all $t \in [0, T]$, and let $\mathcal{G} = \mathcal{G}_T$. It is well-known that $\mathbb{G}$ is the smallest filtration (satisfying the usual conditions) which makes τ a $\mathbb{G}$-stopping time and contains $\mathbb{F}$, in the sense that $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0, T]$.

The price at time $t \in [0, T]$ of one share issued by the defaultable firm is denoted by S_t. We assume that the $\mathbb{G}$-adapted process $S = (S_t)_{0\le t\le T}$ is continuous and strictly positive on the stochastic interval $[\![0, \tau[\![$ and satisfies $S\mathbf{1}_{[\![\tau,T]\!]} = 0$. This means that S drops to zero as soon as the default event occurs and remains thereafter frozen at that level. By relying on Sect. 5.1 of [18] together with the fact that all $\mathbb{F}$-martingales are continuous, it can be proved that there exists a continuous strictly positive $\mathbb{F}$-adapted process $\tilde{S} = (\tilde{S}_t)_{0\le t\le T}$ such that $S_t = \mathbf{1}_{\{\tau>t\}}\tilde{S}_t$ holds for all $t \in [0, T]$. We shall refer to the process $\tilde{S}$ as the *pre-default* value of S.

The pre-default value $\tilde{S}$ is assumed to be influenced by the $\mathbb{F}$-adapted *stochastic volatility* process $v = (v_t)_{0\le t\le T}$ and by an $\mathbb{R}^{d-2}$-valued $\mathbb{F}$-adapted *factor process* $Y = (Y_t)_{0\le t\le T}$. The process Y can include macro-economic covariates describing the state of the economy as well as firm-specific and latent variables, as considered e.g. in [19,20]. Let us define the process $L = (L_t)_{0\le t\le T}$ by $L_t := \log\tilde{S}_t$ and the $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process $X = (X_t)_{0\le t\le T}$ by $X_t := (v_t, Y_t^\top, L_t)^\top$, with $^\top$ denoting transposition.

The processes v, Y and L are jointly specified through the following square-root-type SDE for the process X on the state space $\mathbb{R}^m_{++}\times\mathbb{R}^{d-m}$, where we let $\mathbb{R}^m_{++} := \{x \in \mathbb{R}^m : x_i > 0, \forall i = 1, \dots, m\}$, for some fixed $m \in \{1, \dots, d-1\}$:

$$dX_t = (AX_t + b)\,dt + \Sigma\sqrt{R_t}\,dW_t, \qquad X_0 = \left(v_0, Y_0^\top, \log S_0\right)^\top = \bar{x} \in \mathbb{R}^m_{++}\times\mathbb{R}^{d-m}, \tag{2.1}$$

where $(A, b, \Sigma) \in \mathbb{R}^{d\times d}\times\mathbb{R}^d\times\mathbb{R}^{d\times d}$ and R_t is a diagonal $(d\times d)$-matrix with elements $R_t^{i,i} = \alpha_i + \beta_i^\top X_t$, for all $t \in [0, T]$, with $\alpha := (\alpha_1, \dots, \alpha_d)^\top \in \mathbb{R}^d_+$ and $\beta := (\beta_1, \dots, \beta_d) \in \mathbb{R}^{d\times d}_+$.

Following the notation adopted in Chapt. 10 of [21], for a given $m \in \{1, \dots, d-1\}$, we define the sets $I := \{1, \dots, m\}$, $J := \{m+1, \dots, d\}$ and $D := I \cup J = \{1, \dots, d\}$. Intuitively, the set I collects the indices of the first m elements of the $\mathbb{R}^d$-valued process X, while the set J collects the remaining ones. In order to guarantee the existence of a strong solution to the SDE (2.1), we introduce the following assumption.

Assumption 2.1. The parameters $A, b, \Sigma, \alpha, \beta$ satisfy the following conditions:

(i) $b_i \ge (\Sigma_{i,i})^2\beta_{i,i}/2$ for all $i \in I$;
(ii) $A_{i,j} = 0$ for all $i \in I$ and $j \in J$ and $A_{i,j} \ge 0$ for all $i, j \in I$ with $i \neq j$;
(iii) $\Sigma_{i,j} = 0$ for all $i \in I$ and $j \in D$ with $j \neq i$;
(iv) $\beta_{j,i} = 0$ for all $i \in D$ and $j \in J$, $\beta_{i,i} > 0$ for all $i \in I$ and $\beta_{i,j} = 0$ for all $i, j \in I$ with $i \neq j$;
(v) $\alpha_i = 0$ for all $i \in I$ and $\alpha_j > -\sum_{i\in I}\beta_{i,j}$ for all $j \in J$.

For any $\bar{x} \in \mathbb{R}^m_{++}\times\mathbb{R}^{d-m}$, Assumption 2.1 ensures the existence of a unique strong solution $X = (X_t)_{0\le t\le T}$ to the SDE (2.1) on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, P)$ such that $X_0 = \bar{x}$ and $X_t \in \mathbb{R}^m_{++}\times\mathbb{R}^{d-m}$ P-a.s. for all $t \in [0, T]$. Indeed,

[1] Due to Lemmas 6.1.1 and 6.1.2 of [18], the fact that $P(\tau > t|\mathcal{F}_T) = P(\tau > t|\mathcal{F}_t)$, for all $t \in [0, T]$, implies that all $(P, \mathbb{F})$-martingales are also $(P, \mathbb{G})$-martingales. In particular, $W = (W_t)_{0\le t\le T}$ is a Brownian motion with respect to both $\mathbb{F}$ and $\mathbb{G}$. This important fact will be used in the following without further mention.