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Pricing formulae for constant proportion debt obligation notes: The Laplace transform technique

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ABSTRACT

In this paper we derive closed form pricing formulae for the constant proportion debt obligation (CPDO) by using the Laplace transform technique. First, we present the pricing equation as a combination of a pricing problem (conditional expectation) and a static part that depends only on time. Then, we indicate that the pricing problem is in fact a pricing of a barrier option written on the shortfall. Hence, we derive explicit solutions of such barrier option problems when the shortfall follows either a diffusion or a double exponential jump diffusion process. Finally, we illustrate and discuss the results using numerical applications.

1. Introduction

Constant Proportion Debt Obligation (CPDO) is a structured credit product which is designed for offering a coupon over LIBOR. The structure of CPDO has an important characteristic. CPDO aims to generate these high coupons by taking a leveraged position on a portfolio of credit default swap (CDS) indices [1]. CPDOs are issued by means of a Special Purpose Vehicle (SPV). SPV collects the proceeds from holders of CPDO at issuance and invests them on a bank account. This amount is called "*collateral*". Then, the leveraged position is taken on a notional index portfolio. In industry the notional size of the leveraged position is represented by

$$\ell_t^U = \max\left\{\frac{(F_t - V_t)}{PVS_t} \times m,\ 0\right\}, \tag{1}$$

where F_t is the present value of the coupons and principle, V_t is the current assets of the CPDO manager, PVS_t is the current rate of the portfolio of CDS indexes and m is a constant predetermined multiple. The leveraged position is dynamically adjusted according to mark-to-market gains and losses [2,3].

In CPDO the gap between the present value of future obligations F_t and the current assets V_t of the CPDO manager is called "*shortfall*". The leveraged position is directly affected from the shortfall. If there is no shortfall then the current value of the collateral is said to be enough to cover all the coupons and principles. Then, the leveraged position is closed and the manager only keeps the bank account. This case is referred to as a "*cash-in*" event for CPDOs. Similarly, a "*cash-out*" event for CPDOs is the case in which the shortfall exceeds a given threshold. This case may also be considered as early default [2–4].

The literature on CPDOs is very limited. The first study seems to be Dorn's, published first as a preprint in 2007, then revised and published as an article [4]. Using a structural approach, Dorn found a closed form valuation formula for the CPDO price. Moreover, he investigated the relations between the CPDO and inverse Constant Proportion Portfolio Insurance.

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In order to determine the optimal leverage functions for the CPDOs, Baydar, Di Graziano and Korn [5] used a stochastic control method: optimality is then defined as maximizing the utility obtained from the final payoff of the CPDO at maturity time T. In their study leverage changes continuously and the risky reference portfolio is assumed to follow a Brownian motion with drift.

Later, Çekiç, Korn, and Uğur [2] performed a study on the same problem for the geometric Brownian risky index and made a sensitivity analysis with respect to the parameters of the system. Recently, Cont and Jessen [6] included the modelling of default risk, loss distribution and introduced other risk factors to the problem of CPDOs. They examine all the movements of the credit default swap markets deeply and they provide a rating model for CPDOs.

In 2010, Çekiç, Korn and Uğur [3] implemented a study that includes the minimization of the mean-square distance between the promised principle and the final value of the assets of the CPDO manager by using both the martingale approach and the optimal control method for a geometric Brownian risky index. In the study, they show that the optimal leverage strategy for CPDOs in a mean-square sense coincides with the leverage strategy used in industry.

In this study we emphasize a model for the fair pricing of CPDOs. As observed from the studies performed by risk management agencies and literature about the pricing and rating of CPDOs the models include too many variables. The implementations of the formulae are very challenging in decision mechanisms. We combine the pricing models with the simple definition of the CPDO assets of the manager stated in optimal leverage papers.

The current work is organized as follows. In Section 2, we model the CPDOs as a combination of a barrier option pricing problem and a static part that depends only on time. Then, in Section 3 we obtain the closed form formulae for the fair price of the CPDOs under the Laplace domain. In this part we derive the formulae when the risky portfolio of indexes follows both diffusion and double exponential jump diffusion processes. In Section 4 we make a numerical analysis to observe the changes in the CPDO price over time. Finally, in Section 5 we give the concluding remarks.

2. Modelling the CPDOs

In this part of the paper, we model CPDOs by considering all the risks embedded in CPDOs from the holders' side.

Generally, the CPDO includes two types of risk for a holder. One is a "cash-out" event which is also called the "*early default*". That is,

$$\text{If } V_t \leq \beta_t, \text{ then, early default occurs.} \tag{2}$$

Here, β_t is the predetermined constant or can be regarded as a time-varying barrier. We define a stopping time for this case as

$$\tau_{def} = \inf\{t \in (0, T) \mid V_t < \beta_t\}.$$

The other risk is in guaranteeing the principal. Suppose the CPDO lives until maturity time T, but, the principal payment G is not generated. Such a case is said to have "*default on principle*". In other words, if $P_{CPDO}(v, T)$ denotes the final payment of the CPDO, then

$$P_{CPDO}(v, T) = G - \max\{G - v, 0\} = \begin{cases} v, & \text{if } v \leq G, \\ G, & \text{otherwise.} \end{cases} \tag{3}$$

Thus, the principal is guaranteed only when assets of the CPDO manager are greater than the principle. Otherwise, the assets are distributed to the holders of the CPDO.

In order to model *cash in* event we also define a stopping time τ_{str} for the CPDOs:

$$\tau_{str} = \inf\{t \in (0, T) \mid V_t \geq F_t\}.$$

In calculations we use continuous time trading and we take the wealth definition as stated in [5]. At issuance the CPDO manager pays some initial costs, and therefore, we assume that the initial wealth is $V_0 = v \leq G$. The wealth process is then defined as

$$dV_t = rV_t dt + \ell_t dS_t - F_T(r + \nu)dt, \qquad V_0 = v. \tag{4}$$

In this setting, the first part of (4) shows the deposit account of the CPDO, and r represents the constant risk free interest rate. Moreover in the second part, unlike the definition stated in [6], all the mark-to-market gains and losses caused from the portfolio of credit default swaps including

- the changes in index movements,
- the losses observed because of the defaults of CDOs included in the indexes, and
- the changes observed in the index values caused from rearrangement of the indexes,

are considered with a single element denoted by dS_t. The third part of (4) is the reduction part of the wealth representing coupon payments of CPDOs which are greater than the risk free rate for each unit of money invested by note holders, where ν denotes this surplus return.