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Delta hedging in discrete time under stochastic interest rate

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ABSTRACT

We propose a methodology based on the Laplace transform to compute the variance of the hedging error due to time discretization for financial derivatives when the interest rate is stochastic. Our approach can be applied to any affine model for asset prices and to a very general class of hedging strategies, including Delta hedging. We apply it in a two-dimensional market model, obtained by combining the models of Black–Scholes and Vasicek, where we compare a strategy that correctly takes into account the variability of interest rates to one that erroneously assumes that they are deterministic. We show that the differences between the two strategies can be very significant. The factors with stronger influence are the ratio between the standard deviations of the equity and that of the interest rate, and their correlation. The methodology is also applied to study the Delta hedging strategy for an interest rate option in the Cox–Ingersoll and Ross model, measuring the variance of the hedging error as a function of the frequency of the rebalancing dates. We compare the results obtained to those coming from a classical Monte Carlo simulation.

1. Introduction

Most of the mathematical models for arbitrage pricing in continuous time assume that markets are always open and that trading is performed continuously in time. Although it is obvious that such an assumption does not hold in practice, the pricing formulas and the hedging strategies valid in the case of continuous trading are usually also adopted in everyday practical situations. Our goal is to propose a methodology to evaluate the impact of trading in discrete time when hedging strategies are constructed under a continuous time assumption.

The object of our investigation is the ex-ante assessment of the performances of dynamic trading strategies. Probably the most notable instance of such a problem is measuring the hedging error of a strategy, based on a liquid assets, that tries to hedge a future liability. Problems of such a kind arise when replicating either a claim using futures contracts, or a payoff of a derivative security with a Delta hedging strategy based on the underlying asset, and in any case when a dynamic strategy is adopted. Ex-ante, a possible way to measure the performance of a strategy is by evaluating expected value and variance of its hedging error. This is usually done by approximations or by Monte Carlo simulations. The approach we propose, based on Laplace transforms, allows to efficiently perform such computations for a very general class of models. This paper is the third one of a series of studies that addressed such an issue in different settings. Our previous works on this subject, to whom we refer for a deeper introduction to the problem, are Angelini and Herzel [1,2], the first dealing with market models based on Lèvy processes, the second where the more general class of affine processes are considered.

We consider a market model driven by continuous time affine processes, in which, by definition, the conditional characteristic function is an exponential of an affine function of the state variables (see Duffie et al. [3] for a formal definition

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and properties of affine models). In this framework, Angelini and Herzel [1,2] provide semi-closed formulas for the efficient computation of expected value and variance of the hedging error for a quite general class of strategies, called "affine", that includes the popular Delta hedging strategy. Such formulas are obtained by using a Laplace transform approach, that is based on the idea of writing the payoff of the contingent claim as an inverse Laplace transform, introduced by Hubalek et al. [4] in the context of variance–optimal hedging. An important feature of the result is that one can study different types of misspecification. For instance, it is possible to analyze the performance of the standard Black–Scholes Delta strategy when the underlying asset is driven by a process which is not log-normal, like in a stochastic volatility model.

In our previous contributions we made the simplifying assumption of deterministic interest rates. In the present work, we extend the analysis to the case of stochastic interest rates. Such an extension gives us the opportunity to study the hedging problem in a more general and realistic model. For example, we can measure the effect of assuming that the interest rate is deterministic when in fact it is stochastic. As an example, we consider a simple two-dimensional affine model, where the underlying evolves according to the Black–Scholes dynamics, while the short-term interest rate follows the process of the Vasicek model, and the stock and the interest rate may be correlated. This is a particular case of a model considered in van Haastrecht et al. [5] to price long-term derivatives. Within this model, we implement two types of Delta strategies: the correct strategy that takes into account the randomness of the interest rate, which may be called the model Delta, and the plain Black–Scholes Delta with deterministic rate. We show that the differences between the two strategies may be very relevant. The most important factors are the correlation and the ratio between the volatility of the risky asset and that of the interest rate. Therefore, the standard Black–Scholes Delta-hedging strategy, still widely used by practitioners, may be not appropriate because it may lead to a variance of the error much higher, in relative terms, to that produced by the correct Delta, especially when the volatility of the interest rates is comparable to that of the stock. It is noteworthy to observe that the relatively poor performances of the Black–Scholes Delta are peculiar in the present setting. In fact, Angelini and Herzel [2] showed that if the interest rates are deterministic but the volatility is stochastic, then the Black–Scholes Delta often outperforms the model Delta. We conclude with a study of the Delta hedging for an interest rate option in the Cox, Ingersoll and Ross model [6], providing numerical illustrations for the cases of objective measures different from the risk-neutral measures used to implement the strategy. In that setting we are also able to provide a further numerical validation of the precision of our algorithm, by comparing its results to those obtained by simulations.

2. The computational algorithm

Let us consider the problem of hedging a European contingent claim with maturity T, whose payoff H is represented as an inverse Laplace transform:

$$H = \int_{\mathcal{C}} e^{zy_T} \Pi(dz), \tag{2.1}$$

where $\mathcal{C} = R + i\mathbb{R}$, with $R \in \mathbb{R}$, Π is a finite complex measure on $\mathcal{C}$ and $y_T = \ln(S_T)$, where S is the price of a risky asset. The log-return $y = \ln(S)$ of the underlying asset and a short term stochastic interest rate r are components of a multi-dimensional affine process X, whose other components may include stochastic volatility, dividend yields, etc. The simplest example of such a model is obtained by taking the Black–Scholes dynamics for the underlying and a short rate model for the interest rate, like the Vasicek model [7]. In this case one can also consider a non zero correlation between stock and interest rate. We will use this model for applications in Section 3.1. If the Cox, Ingersoll and Ross model [6] is used for the interest rate, the resulting two-dimensional model would be affine if and only if the correlation is zero. A model that includes stochastic volatility as well as stochastic interest rate is studied in van Haastrecht et al. [5]. Pan [8] studied a four-dimensional affine model combining stochastic volatility, interest rates and dividend yield.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t<\infty}, P)$ be a filtered probability space satisfying the usual technical conditions. We interpret P as the physical or objective probability measure. Let us consider an affine time-homogeneous Markov process X defined in a state space $D \subset \mathbb{R}^d$ and write its conditional characteristic function as

$$\phi(u, X_t, t, s) = E_t\left[e^{u\cdot X_s}\right] = e^{\alpha(u,t,s)+\beta(u,t,s)\cdot X_t}, \tag{2.2}$$

where $u \in i\mathbb{R}^d, t, s \in [0, T]$ with $t \le s, E_t$ denotes the expected value conditional on $\mathcal{F}_t$ and $\cdot$ the scalar product. The functions $\alpha(u, t, s)$ and $\beta(u, t, s)$ go from $i\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{C}$ and to $\mathbb{C}^d$ respectively, and satisfy a system of Riccati equations whose general form is given in Duffie, Pan and Singleton [3, Eqs. (2.5) and (2.6)]. We suppose that the functions $\alpha(u, t, T)$ and $\beta(u, t, T)$ can be analytically extended to an open convex domain U containing $0 \in \mathbb{C}^d$ for all $t \in [0, T]$. In this paper we skip technical conditions on the domain U (see Angelini and Herzel [2] for a thorough analysis on this point).

We also assume that X is affine under a pricing measure Q. Conditions for a process to be affine under both measures P and Q are given by Duffie, Pan and Singleton [3]. We consider the discounted conditional characteristic function

$$\begin{aligned}\psi(u, X_t, t, s) &= E_t^Q\left[\exp\left(-\int_t^s r_\tau d\tau\right) e^{u\cdot X_s}\right] \\ &= e^{\tilde{\alpha}(u,t,s)+\tilde{\beta}(u,t,s)\cdot X_t}.\end{aligned} \tag{2.3}$$