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Mean semi-deviation from a target and robust portfolio choice under distribution and mean return ambiguity

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ABSTRACT

We consider the problem of optimal portfolio choice using the lower partial moments risk measure for a market consisting of n risky assets and a riskless asset. For when the mean return vector and variance/covariance matrix of the risky assets are specified without specifying a return distribution, we derive distributionally robust portfolio rules. We then address potential uncertainty (ambiguity) in the mean return vector as well, in addition to distribution ambiguity, and derive a closed-form portfolio rule for when the uncertainty in the return vector is modelled via an ellipsoidal uncertainty set. Our result also indicates a choice criterion for the radius of ambiguity of the ellipsoid. Using the adjustable robustness paradigm we extend the single-period results to multiple periods, and derive closed-form dynamic portfolio policies which mimic closely the single-period policy.

1. Introduction

The purpose of this paper is to give an explicit solution to the optimal portfolio choice problem by minimizing the lower partial moment risk measure of mean semi-deviation from a target return under distribution and mean return ambiguity using a robust optimization (RO) approach.

Portfolio optimization in single and multiple periods, using different criteria such as mean–variance and utility functions, has been studied extensively; see, e.g., [1–14]. In particular, Hakansson [5] treats correlations between time periods while Merton [8,9,15] concentrates on continuous-time problems. These references usually consider a stochastic model for the uncertain elements (asset returns) and study the properties of an optimal portfolio policy. An important tool here is stochastic dynamic programming.

The philosophy of robust optimization (RO) [16,17] is to treat the uncertain parameters in an optimization problem by confining their values to some uncertainty set without defining a stochastic model, and find a solution that satisfies the constraints of the problem regardless of the realization of the uncertain parameters in the uncertainty set. It has been applied with success to single-period portfolio optimization; see, e.g., [18–21]. The usual approach is to choose uncertainty sets that lead to tractable convex programming problems that are solved numerically. In the present paper, we instead find closed-form portfolio rules. In the case of multiple-period portfolio problems, RO was extended to adjustable robust optimization (ARO), an approach that does not resort to dynamic programming, and is more flexible than the classical RO for sequential problems, but may lead to more difficult optimization problem instances; see [22,23]. A related approach, which is data-driven with probabilistic guarantees and scenario generation, is explored in e.g. [24].

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The optimal portfolio choice problem using lower partial moments risk measures under distribution ambiguity was studied by Chen, He and Zhang in a recent paper [25] in the case of n risky assets. The authors assumed that the mean return vector μ and variance–covariance matrix Γ of risky assets are fixed, and compute portfolios that are distributionally robust in the sense that they minimize a worst-case lower partial moment risk measure over all distributions with fixed first-moment and second-moment information. They obtained closed-form distributionally robust optimal portfolio rules. In the present paper we first extend their result to the case where a riskless asset is also included in the asset universe, a case which is an integral part of optimal portfolio choice theory. The inclusion of the riskless asset in the asset universe simplifies considerably the optimal choice formula in some cases as we shall see below in Theorem 1. A criticism levelled against the distributionally robust portfolios of Chen et al. [25] is the sensitivity of these portfolios to uncertainties or estimation errors in the mean return data, a case that we refer to as *mean return ambiguity*; see [18]. To address this issue, we analyse the problem for when the mean return is subject to ellipsoidal uncertainty in addition to distribution ambiguity and derive a closed-form portfolio rule. Since the majority of contributions in robust portfolio optimization aim at providing convex optimization formulations our explicit portfolio rule constitutes a worthy addition to the literature. Our result is valid for choices of the ellipsoidal uncertainty (ambiguity radius) parameter ϵ not exceeding the optimal Sharpe ratio attainable in the market. Furthermore, the difference between the optimal mean semi-deviation risk under distribution ambiguity only and the same measure under joint uncertainty in distribution and mean return may also impose an optimal choice of ϵ, an observation which we illustrate numerically. For other related studies on portfolio optimization with distributional robustness, the reader is referred to [19,26,27]. We also obtain optimal dynamic portfolio rules using the adjustable robust optimization paradigm [22,23] for both cases of distribution ambiguity and expected return ambiguity combined with distribution ambiguity. The resulting portfolio rules are myopic replicas of the single-period results. The plan of the paper is as follows. In Section 2 we derive the optimal portfolio rules under distributional ambiguity for two measures of risk in the presence of a riskless asset. We study the multiple-period adjustable robust portfolio rules in Section 3. In Section 4, we derive the optimal portfolio rule for the mean squared semi-deviation from a target measure under distributional ambiguity and ellipsoidal mean return uncertainty. We also discuss the optimal choice of the uncertainty/ambiguity radius for the mean return. The multiple-period extension is given in Section 5.

2. Minimizing lower partial moments in the presence of a riskless asset: single period

The lower partial moment risk measure LPM_m for $m = 0, 1, 2$ is defined as

$$\mathbb{E}\,[r - X]_+^m$$

for a random variable X and target r. We assume, in addition to the n risky assets with given mean return μ and variance–covariance matrix Γ, that a riskless asset with return rate $R < r$ exists. If $R \geq r$, then the benchmark rate is attained without risk, i.e. the lower partial moment LPM_m is minimized taking value 0 by investing entirely in the riskless asset. Denote by y the variable for the riskless asset, for handling it separately, and by e the n-dimensional vector of entries 1; the LPM_m minimizing robust portfolio selection model under distribution ambiguity is

$$RPR_m = \min_{x,y} \sup_{\xi\sim(\mu,\Gamma)} \mathbb{E}\left[r - x^T\xi - yR\right]_+^m \tag{1}$$

$$\text{s.t}\; x^Te + y = 1. \tag{2}$$

We use the notation $\xi \sim (\mu, \Gamma)$ to mean that random vector ξ belongs to the set whose elements have mean μ and variance–covariance matrix Γ. Now, we provide the analytical solutions of the riskless asset counterpart of the problem for $m = 1, 2$ (expected shortfall and expected squared semi-deviation from a target, respectively) following a similar line to the proof of LPM_m solutions in [25]. The optimal portfolio choice for $m = 0$, which corresponds to minimizing the probability of falling short of the target, is uninteresting in the presence of a riskless asset in comparison to the case of risky assets only, since the optimal portfolio displays an extreme behaviour (the components vanish or go to infinity). Therefore, we exclude this case in the theorem below.

Theorem 1. *Suppose $\Gamma \succ 0$ and $R < r$. The optimal portfolio in (1)–(2) is obtained in the two different cases as follows.*

1. *For the case $m = 1$ the optimal portfolio rule is*

$$x^* = \frac{2\tilde{r}}{1+H}\Gamma^{-1}\tilde{\mu}.$$

2. *For the case $m = 2$ the optimal portfolio rule is*

$$x^* = \frac{\tilde{r}}{1+H}\Gamma^{-1}\tilde{\mu},$$

where $H = \tilde{\mu}^T\Gamma^{-1}\tilde{\mu}$, $\tilde{\mu} = \mu - Re$ and $\tilde{r} = r - R$.