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Optimal control of stochastic hybrid system with jumps: A numerical approximation

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ABSTRACT

The generalized class of stochastic hybrid systems consists of models with regime changes including the occurrence of impulsive behavior. In this paper, the stochastic hybrid processes with jumps are approximated by locally consistent Markov decision processes that preserve local mean and covariance. We further apply a randomized switching policy for approximating the dynamics on the switching boundaries. To investigate the validity of the approximation, we study a stochastic optimal control problem. On the basis of the discrete approximation, we solve the optimal control problem and show that the solution obtained from the discretized problem converges to the solution of the original optimal control problem.

1. Introduction

Stochastic hybrid systems (SHS) are dynamical systems with interacting continuous and discrete behaviors. This generalized class of systems arises in numerous applications of dynamical models with multiple modes. Engineering systems like air traffic management, biological networks and manufacturing systems are characterized by hybrid dynamics as they exhibit complex behaviors resulting from the interactions between heterogeneous components. The analysis and control of a stochastic hybrid system have attracted considerable interest from many researchers and constitute an important topic in hybrid system modeling. For continuous SHS, early contributions include the studies of [1–4], with [5,6] establishing theoretical foundations for the measurability of events for reachability problems. In [7], the authors focus on the reachability problem using the Markov chain approximation of [8] and apply the results to air traffic management problems. For various applications of stochastic hybrid systems we refer the reader to [9,10] and the references therein. Some other classes of stochastic hybrid systems which have been studied in the literature include diffusion processes with Markovian switching parameters [11,12], switching diffusions [1], and Markov decision drift processes [13].

In this paper, we concentrate on the approximation of SHS, where the continuous state is governed by a stochastic differential equation of Itô–Skorohod type with the drift coefficient, the diffusion matrix and a *jump* function depending on the discrete component. The main difference of our model from related SHS models is the presence of controlled jumps which represent the impulsive behavior in the continuous state. We apply the well-known approach presented in [8] to our extended SHS which incorporates jumps in the continuous state with autonomous switchings in the discrete state. The approach has been considered in many different classes of stochastic systems [14–16]; however, the extension to SHS with jumps (*SHSJ*) remains novel. The main contribution of this paper is the application of the weak approximation approach of [8] to controlled autonomous SHS with jumps, which is extended to autonomously switching controlled diffusions by [15] and to SHS by [14]. We extend the method employed in [14] to SHSJ where a randomized switching policy that guarantees the

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continuity of switching times under certain conditions is introduced. Then, using an extension of the methodology presented in [8], we show that the approximating processes converge to the original stochastic hybrid systems.

The approximation of SHS is achieved by appropriately chosen Markov decision processes (MDPs) defined on a discrete state space. The discrete approximating processes are chosen in such a way that they preserve certain "local" properties that are similar to those of the original controlled process. Then, to investigate the validity of the approximation we consider the stochastic optimal control problem of minimizing a cost until a target set is reached. The advantage of the discretization method is that the cost of the approximating MDPs converges weakly to the original cost. Under rather broad conditions, we also prove that the solution of the problem for the sequence of the approximating processes converges to that for the underlying original process as the approximation parameter goes to zero.

The rest of the paper is organized as follows. Section 2 introduces our SHSJ model and Section 3 presents the stochastic optimal control problem as well as the approximation method. In Section 4, computational methods and the discretized optimal control problem are described. Convergence results and optimality discussions are presented in Section 5. A numerical example from finance is provided to illustrate the performance of the approximation method in Section 6. Finally, some additional remarks and ideas for future studies are provided in Section 7.

2. Stochastic hybrid systems with jumps

We define a stochastic hybrid system with jumps as follows:

$$(X, \Theta, U, \Omega, A, a, b, f, \delta, R, (x_0, \theta_0)), \tag{1}$$

where

(i) $X \subseteq \mathbb{R}^d$ is the continuous state space,
(ii) $\Theta = \{1, 2, \ldots, M\}$ is a finite set of discrete states,
(iii) $U = \{U_\theta\}_{\theta\in\Theta}$, with $U_\theta \subseteq \mathbb{R}^{m_\theta}$ $(\theta \in \Theta)$, is a finite family of sets of continuous controls,
(iv) the partition of X is denoted by $\Omega = \{\Omega_\theta\}_{\theta\in\Theta}$, with $\Omega_\theta \subseteq \mathbb{R}^d$ $(\theta \in \Theta)$,
(v) $A = \{A_\theta\}_{\theta\in\Theta}$, with $A_\theta \subset \partial\Omega_\theta$ $(\theta \in \Theta)$, is the collection of switching points,
(vi) $a : \mathbb{R}^d \times \Theta \times U_\theta \to \mathbb{R}^d$ is a controlled drift term,
(vii) $b : \mathbb{R}^d \times \Theta \times U_\theta \to \mathbb{R}^d \times \mathbb{R}^d$ is a controlled diffusion term,
(viii) $f : \mathbb{R}^d \times \Theta \times U_\theta \times \mathbb{R} \to \mathbb{R}$ is a controlled jump function,
(ix) $\delta : \Theta \times A \to \Theta$ is an autonomous switching map,
(x) $R : \Theta \times A \to P(X)$ is the reset map which assigns a reset probability kernel to each $x(\cdot) \in A_\theta$ and θ on X, and $P(X)$ is a family of probability measures on X,
(xi) (x_0, θ_0) is the initial state defined on $X \times \Theta$ at time $t_0 = 0$.

To describe the dynamics of an SHSJ, we need to consider an $\mathbb{R}^d$-valued standard Wiener process $W(\cdot)$ and a Poisson random measure $\overline{N}(\cdot, \cdot)$. Let us suppose that at time t_i $(i \in \mathbb{N}_0)$, the state of the SHSJ is $(x_i, \theta_i) = \left(x_{t_i}, \theta_{t_i}\right)$ with $x_i \in \Omega^0_{\theta_i}$. Here, Ω^0 denotes the interior of the set Ω. While the continuous state x_t stays in $\Omega^0_{\theta_i}$ and the discrete state θ_t remains constant, the evolution of x_t is given by the following stochastic differential equation (SDE):

$$dx_t = a(x_t, \theta_t, u)\, dt + b(x_t, \theta_t, u)\, dW_t + \int_{\mathbb{R}} f(x_t, \theta_t, u, y)\, \overline{N}(dt, dy), \tag{2}$$

where $\overline{N}(dt, dy)$ is defined as

$$\overline{N}(dt, dy) = \begin{cases} N(dt, dy) - \nu(dy)dt, & \text{if } |y| < R, \\ N(dt, dy), & \text{if } |y| < R, \end{cases}$$

for some $R \in [0, \infty]$. Here, $N(\cdot, \cdot)$ denotes a Poisson random measure with intensity $dt \times \nu(dy)$, where $\nu(\cdot)$ is the Lévy measure of x_t on a compact support $\Gamma \subset \mathbb{R}$.

Let us define the stopping time $t_{i+1} := \inf\{t > t_i : x_t \notin \Omega^0_{q_i}\}$. At time t_{i+1}, the discrete state switches autonomously, resulting in a reset in the continuous state. Following this switching, the new discrete state becomes $\theta_{i+1} = \delta(\theta_i, x(t^-_{i+1}))$ while the new continuous state $x_{t_{i+1}}$ is randomly determined according to the probability measure $\mathbb{P}(\theta_i, x_{t^-_{i+1}})(\Xi)$, where $\Xi \subseteq \Omega_{\theta_{i+1}} \subseteq \mathbb{R}^d$ is a measurable set. Here, $x(t^-)$ denotes the left limit of $x(t)$. The evolution of $x(t)$ is then described by the SDE (2) with $\theta(t) = \theta_{i+1}$ and initial condition $x(t_{i+1})$ until the next switching time. Thus, x_t switches from one jump diffusion path to another, as the discrete state θ_t moves from one state to another.

With these definition and this notation, a typical stochastic execution starts at time t_0, from (x_0, θ_0), and the continuous state x_t evolves according to the SDE (2) until time t_1 when x_t first hits $\partial\Omega_{\theta_0}$. Then, depending on the hitting position $x(t^-_1)$, the discrete state jumps to $\theta(t_1) = \theta_1$ and the continuous state is reset randomly to $x(t_1) = x_1$.

Considering the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we assume that the functions $a(x, \theta, u)$, $b(x, \theta, u)$ and $f(x, \theta, u, y)$ satisfy Lipschitz and growth conditions, for a given $\theta \in \Theta$. Moreover, we assume that a, b and f are continuous with respect to (x, θ, u), and $f(x, \theta, u, \cdot)$ is bounded uniformly in $u \in U$, in a neighborhood of $y = 0$. These assumptions ensure that for