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Pricing and hedging of inflation-indexed bonds in an affine framework

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ABSTRACT

This study deals with the pricing and hedging of inflation-indexed bonds. Under foreign exchange analogy we model the nominal short rate, real short rate and logarithm of the price index with an affine Gaussian process. Using the underlying affine property, we compute the nominal and inflation-indexed bond prices explicitly. We derive no-arbitrage drift conditions for the factor process. Then, we perform a novel hedging analysis where our objective is to replicate an indexed bond of a given maturity by trading a portfolio of nominal bonds. This analysis leads to a hedging criterion based on a set of restrictions on the eigenvalues and the eigenvectors of mean reversion speed matrix of the factor process. We fit the model to the US bond data and perform an in-sample hedging analysis. Having relatively small in-sample hedging errors, we validate the theoretical hedging result for the considered dataset.

1. Introduction

During the last decade total outstanding volume of inflation-indexed bonds has grown strongly in accordance with the rising demand from the investors who need protection against inflation-linked liabilities. Unlike a regular nominal bond, an inflation-indexed bond guarantees the real interest rate regardless of the future realised inflation, provided that it is held until its maturity. In other words, fixing the payments in real terms, an indexed-bond protects both investors and issuers against the inflation risk.

The foreign exchange analogy is the most widely used approach for the modelling of inflation-indexed bonds. The rationale of this approach is based on the idea of specifying the real and nominal interest rate term structures and considering the nominal and real parts of the economy as the domestic and foreign economies, respectively. Accordingly, it becomes possible to treat the price index process as an exchange rate between the real and nominal economies. The first pricing model for inflation-indexed products is proposed by Hughston [1], where a foreign exchange analogy is used. In order to price indexed-bonds, [2] models the evolution of the nominal and real term structures and the consumer price index by using an HJM (see, [3]) methodology. As an extension of this study, [4] proposes a pricing model in which nominal and real forward interest rates and the consumer price index are allowed to be driven by a standard Brownian motion and a general marked point process. Considering a three-factor Gaussian method, Kjaergaard [5] models inflation dynamics and gives closed-form expressions for the index and discount factors. An alternative approach is proposed independently by Belgrade et al. [6] and Mercurio [7] in which the Libor market model is adapted to inflation modelling.

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In this study, under foreign exchange analogy we model the nominal short rate, real short rate and logarithm of the price index with an affine Gaussian process. More specifically, we propose a short-rate model which allows for the long run as well as instantaneous relations between the real rate, nominal rate and inflation. Under this modelling setup, first we use the theory of affine processes and obtain closed-form nominal and inflation-indexed bond prices. Then, we impose no-arbitrage assumption and find the resulting drift restrictions on the factor process. In particular, one of these restrictions corresponds to the well-known Fisher equation. Second, this study deals with a novel hedging problem. We investigate conditions under which an inflation-indexed bond of a given maturity can be replicated by dynamically trading on a portfolio of nominal bonds. As the main contribution of this study, we provide a hedging criterion based on a set of restrictions on the eigenvalues and the eigenvectors of mean reversion speed matrix of the factor process. Finally, in order to test the theoretical hedging result we estimate the model and perform an in-sample hedging analysis on the US bond market data. Overall, we validate the main hedging result for the considered dataset.

The remaining of this paper is structured as follows. In Section 2, we introduce the underlying modelling framework. In the following section, we present no-arbitrage drift restrictions. In Section 4, we derive the pricing formulae for the nominal and inflation-indexed bonds. We discuss the hedging problem and give the main theoretical result in Section 5. In Section 6, we provide the application methodology in detail and give the empirical estimation and hedging results. Finally, concluding remarks are stated in Section 7. The proof of the main hedging theorem is given in Appendix A.

2. Modelling framework

From now on, we use the notation $A^\top$ and $|A|$ to denote the transpose and the determinant of the matrix A, respectively. $A^{(i,j)}$ and $A^{(i)}$ address the (i,j)th entry and the ith column, correspondingly. We occasionally denote a matrix with columns a_i by $(a_1|a_2|\cdots|a_n)$, and e_n represents nth basis vector. We consider a finite time horizon $[0, T^*]$ and a frictionless market. The uncertainty in the market is represented by the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions, where $\mathbb{P}$ denotes the historical probability measure. We define, for any $T \le T^*$, the zero-coupon nominal bond which pays one unit of cash at maturity T. The price of the nominal bond at time $t \le T$ is denoted by $P(t,T)$. On the other hand, a zero-coupon inflation-indexed bond is defined as an instrument which pays the nominal value of one unit of price index at time T. Its price is denoted by $\Pi(t,T)$.

Nominal and inflation-indexed bond prices are driven by the nominal short rate, real short rate and the logarithm of the price index, which are represented by the factor process $(X_t)_{t\ge0} = (r_t, \rho_t, \log(I_t))^\top_{t\ge0}$. Under the physical measure $\mathbb{P}$ we assume the following Gaussian dynamics:

$$dX_t = (B^{\mathbb{P}} + \beta^{\mathbb{P}} X_t)dt + \Sigma dW_t^{\mathbb{P}}, \quad X_0 = x \in \mathbb{R}^3, \tag{1}$$

where the column vector $B^{\mathbb{P}} \in \mathbb{R}^3$, the matrices $\beta^{\mathbb{P}}, \Sigma \in \mathbb{R}^{3\times3}$ and $W^{\mathbb{P}}$ is a 3-dimensional standard $\mathbb{P}$-Brownian motion. Given the dynamics for nominal and real short rate processes we can immediately define, for all $t \in [0, T^*]$, the nominal and real saving account processes as $S_t = e^{\int_0^t r_s ds}$ and $S_t^{real} = e^{\int_0^t \rho_s ds}$, respectively.

The theory of affine processes suggests that given the dynamics in (1), X is an affine Gaussian process (for detailed information on affine processes see e.g, [8, Chapter 10]). Affine processes are widely used in finance due to their analytical tractability. Furthermore, in most cases affine factor models yield closed-form bond pricing formulae while for others this class of processes makes it possible to compute prices numerically. In addition to the simplicity it provides for the computation of explicit pricing formulae, this particular choice for the dynamics of the factor process is to provide an extensive hedging analysis that will be given in a later section. The factor process dynamics given in (1) allows for the long-run as well as instantaneous relations between the real rate, nominal rate and inflation. More precisely, the instantaneous and long-run relations between the components of the factor process can be arranged via the form of the matrices Σ and $\beta^{\mathbb{P}}$, respectively.

One of the main objectives of this study is to use affine processes theory to obtain closed-form nominal and inflation-indexed bond prices. To this end, the first task is to write down the dynamics given in (1) under an equivalent martingale measure $\mathbb{Q}$. In order to preserve the affine structure of the factor process under an equivalent change of measure we follow [9] and specify the market price of risk process λ_t as an affine function of the state vector X_t:

$$\lambda_t = \lambda_1 + \lambda_2 X_t,$$

where the column vector $\lambda_1 \in \mathbb{R}^3$ and the matrix $\lambda_2 \in \mathbb{R}^{3\times3}$. Then $W_t = W_t^{\mathbb{P}} + \int_0^t \lambda_s ds$ becomes a Brownian motion under $\mathbb{Q}$ with the Radon–Nikodym density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^{T^*} \lambda_s^\top dW_s - \frac{1}{2}\int_0^{T^*} \|\lambda_s\|^2 ds\right).$$

The dynamics of the factor process under $\mathbb{Q}$ reads

$$dX_t = (B + \beta X_t)dt + \Sigma dW_t$$

with $B = B^{\mathbb{P}} - \Sigma\lambda_1$ and $\beta = \beta^{\mathbb{P}} - \Sigma\lambda_2$.