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Solution examples of an impulse control problem

Rim Amami [a,*], Monique Pontier [b]

[a] Ecole Nationale d'Ingénieurs de Monastir, Tunisia
[b] Institut Mathématique de Toulouse, France

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ABSTRACT

We study an impulse control problem with switching technology in infinite horizon. We solve the impulse control problem in case of deterministic impulse times on specific transition kernel examples. Our objective is to exhibit an optimal strategy which maximizes the value function in these cases.

We study an impulse control problem with switching technology where the control problem variable has three components: impulse time, new technology times and jump law. Specifically, we have an increasing sequence of stopping times $(\tau_n)_{n\geq -1}$ with $\tau_{-1} = 0$, the technology choice ζ_{n+1} at time τ_n and the log firm value jump size Δ_n at time τ_n, with $\Delta_{-1} = 0$. The conditional law of the couple (ζ_{n+1}, Δ_n) depends only on the system status in τ_n^-. Hence, the control has the form $\alpha = (\tau_n, \zeta_{n+1}, \Delta_n, n \geq -1)$.

Optimal impulse control problems were introduced by Bensoussan and Lions [1] and then formalized by other authors. In the past decades, many studies related to this problem have been reported in [2–5]. These studies showed the existence of an optimal strategy but it seems that an explicit form of this optimal strategy is often missing. Barles showed in [6] that the optimal cost function of a deterministic impulse control problem is the unique viscosity solution of a first-order Hamilton–Jacobi quasi-variational inequality. Davis considered in [7] the deterministic optimal control problem. He gave a simple formulation of the dynamic programming principle for piecewise-deterministic Markov processes (PDMPs) which is adequate to solve this kind of problems. Dorobantu et al. assumed in [8] that the activities of the firm are unchanged by the financial structure, which causes debts. In such a case, the debt is perpetual and pays a constant coupon per instant. Thus the optimal control problem is stated with respect to the (coupon policy, default time) pair. The authors showed that the value of the optimal coupon policy decreases if the strict priority rule is removed. Pham, Mnif and Vath [9] studied a financial model with one risk-free and one risky asset subject to liquidity risk and price impact. The authors maximized the expected utility from terminal liquidation value over a finite horizon. This is considered as an impulse control problem under state constraints. The main result of [9] is to characterize the value function as the unique viscosity solution to the associated quasi-variational Hamilton–Jacobi–Bellman inequality. We also refer the reader to Djehiche et al. [10] and Jeanblanc and Hamadène [11] who used purely probabilistic tools as Snell envelop and Backward Stochastic Differential Equations to solve the optimal switching problem in finite horizon. Mnif et al. [12] used the dynamic programming Hamilton–Jacobi–Bellman

* Corresponding author. Tel.: +216 97431048.
E-mail addresses: rim.amami@ymail.com (R. Amami), monique.pontier@math.univ-toulouse.fr (M. Pontier).

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equation of the impulse control problem. The authors provided a numerical scheme for the quasi variational inequality associated to the impulse control problem. Vath and Mnif illustrated in [13] that the impulse control problem is reduced to an iterative sequence of optimal stopping problems. Indeed, the value function is obtained as the limit of an iterative sequence of optimal stopping problems, by using the variational inequalities system. Then, the authors solved this problem numerically using the Monte Carlo method and Malliavin calculus. We mention the studies of [14–17] among others concerning the literature of impulse control problems.

In [18], we built a trajectorial model and proved the existence of an optimal control. In this paper, we simplify the model to get an explicit optimal solution. We will formally assume that the impulse times are deterministic: there exists $t_0 > 0$ such that the impulse times are $\tau_n = (n+1)t_0,\ n \geq -1$.

We consider (ξ_t) the process equal to ζ_{n+1} if $t \in [\tau_n, \tau_{n+1}[$, and (Y_t) the process modelling the firm log value, the jump of Y at time τ_n being Δ_n. This jump could be interpreted as a change decided by the firm council: selling or buying securities or other which increases ($\Delta_n \geq 0$) or decreases ($\Delta_n \leq 0$) the firm value.

The firm net profit is represented by a function f, the switching technology cost is represented by a function c, and $\beta > 0$ is a discount coefficient. To use a strategy $\alpha = ((n+1)t_0, \zeta_{n+1}, \Delta_n)$ leads to a firm profit is defined as

$$k(\alpha) := \int_0^{+\infty} e^{-\beta s} f(\xi_s, Y_s)\, ds - \sum_{n\geq 0} e^{-\beta \tau_n} c(\zeta_n, Y_{\tau_n^-}, \zeta_{n+1}, Y_{\tau_n}). \tag{1}$$

Thus the aim is to find an optimal strategy $\widehat{\alpha}$ which maximizes the value function expectation of $k(\alpha)$. Below in some specific examples of functions f and c, we solve the optimization problem and establish the optimal expected profit of the firm.

This paper is organized as follows: Section 1 is devoted to formulate the impulse control problem and describe the corresponding model. In Section 2, we assume that the conditional laws of the jumps at times τ_n are a set of Gaussian laws, and we specify the gain function f and the cost function c. In Section 3, we assume that the conditional laws of the jumps at times τ_n are a set of exponential laws and we choose linear gain function and quadratic cost function. In both cases we look for an optimal jump law and provide the optimal value. We conclude in Section 4.

1. The model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space with a right continuous complete filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$, and a Brownian motion $W = (W_t)_{t\geq 0}$. Let $(\mathcal{G}_t)_{t>0}$ be the filtration defined by $\mathcal{G}_t = \vee_{s<t} \mathcal{F}_s,\ \forall\, s < t$, (usually denoted by $\mathcal{F}_{t^-}$).

This work is a particular example of the model in [18] and uses some of its results. Let $U = \{0, 1\}$ be the space of possible technologies with 0 being the old technology and 1 the new technology and $\mathcal{P}(U)$ be the trivial sigma-algebra. We assume the impulse times to be deterministic: there exists $t_0 > 0$ such that the impulse times are $\tau_n = (n+1)t_0,\ n \geq -1$. Let ξ be the process modelling the technology state and assume that ξ is right continuous left limited (below denoted as RCLL, for instance see [19]), obviously it takes its values in the set U. Namely it could be written as:

$$\xi_t = \xi_0\, 1_{[0,\tau_0[}(t) + \sum_{n\geq 0} \zeta_{n+1}\, 1_{[\tau_n,\tau_{n+1}[}(t). \tag{2}$$

For every $n \in \mathbb{N}$, ζ_{n+1} the technology choice at time τ_n, is a $\mathcal{G}_{\tau_n}$-measurable random variable. The firm value is defined by $S_t = \exp Y_t,\ t \geq 0$, where Y is the RCLL process defined as follows:

$$Y_t = x + \sum_{n\geq 0} \Delta_n\, 1_{[\tau_n,\tau_{n+1}[}(t) + \int_0^t \left(b(\xi_s)\, ds + \sigma(\xi_s)\, dW_s\right), \tag{3}$$

where Δ_n is too a $\mathcal{G}_{\tau_n}$-measurable random variable, $W = (W_t)_{t\geq 0}$ is a Brownian motion and $b : U \to \mathbb{R}$ and $\sigma : U \to \mathbb{R}^+$ are two functions on U such that $b(1) > b(0)$ and $\sigma(1) > \sigma(0)$.

Definition 1.1. An impulse control is defined by the pair $\alpha = (t_0, r)$, the impulse times being ($\tau_n = (n+1)t_0,\ n \geq -1$) and r being the transition kernel on $U \times \mathbb{R}$, modelling the $\mathcal{G}_{\tau_n}$-conditional law of the couple (ζ_{n+1}, Δ_n):

$$\mathbb{P}(\zeta_{n+1} = j, Y_{\tau_n} = x + dy \mid \zeta_n = i, Y_{\tau_n^-} = x) = r(i, x; j, dy).$$

Thus, for every $n\zeta_{n+1}$ is the technology choice at time τ_n and $\Delta_n = Y_{\tau_n} - Y_{\tau_n^-}$ and recall $\Delta_{-1} = 0$. For each control α, the profit is given by $k(\alpha)$ which is defined above (1). We now specify the profit and cost functions: $f : U \times \mathbb{R} \to \mathbb{R}^+$ is the firm net profit and $c : U \times \mathbb{R} \times U \times \mathbb{R} \to \mathbb{R}^+$ the switching technology cost; they are two non negative Borel functions. The function c satisfies: for every $(i, x) \in U \times \mathbb{R}$, $c(i, x, i, x) = 0$, meaning that the cost is null if there is no switching. Finally, $\beta > 0$ is a discount coefficient.