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An all-at-once approach for the optimal control of the unsteady Burgers equation

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ABSTRACT

We apply an all-at-once method for the optimal control of the unsteady Burgers equation. The nonlinear Burgers equation is discretized in time using the semi-implicit discretization and the resulting first order optimality conditions are solved iteratively by Newton's method. The discretize then optimize approach is used, because it leads to a symmetric indefinite saddle point problem. Numerical results for the distributed unconstrained and control constrained problems illustrate the performance of the all-at-once approach with semi-implicit time discretization.

1. Introduction

Analysis and numerical approximation of optimal control problems (OCPs) for the Burgers equation are important for the development of numerical methods for the optimal control of more complicated models in fluid dynamics such as the Navier–Stokes equations. In the literature, the OCP problems with the Burgers equation are solved either by using the Lagrange–SQP methods [1,2] or by Newton–Lagrange methods [3]. In both approaches, the nonlinear OCP problem is solved as a sequence of linearized convex quadratic OCP problems. A different approach is followed in [4] by transforming the OCP problem into an elliptic equation in space and time and solving the optimality system using COMSOL Multiphysics. In contrast to linear parabolic control problems, the optimal control of the Burgers equation is a non-convex global optimization problem with multiple local minima. Numerical methods can find local solutions close to the starting points. We are interested in local solutions as given in [1] using the Lagrange–SQP method and at each Newton iteration step we solve a quadratic convex optimization problem. Therefore, we consider only first order optimality conditions.

In this paper, we apply the so-called all-at-once method to the optimal control of the time-dependent Burgers equation. In this method, the control and state are treated as independent optimization variables; the optimization problem is explicitly constrained. The optimality system consisting of the state, control and adjoint state variables is solved for all time-steps at once. The discretization of the first order optimality conditions leads to a saddle point system of the form

$$\underbrace{\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}}_{\mathcal{A}} x = b, \tag{1}$$

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where $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite or positive semi-definite and $B \in \mathbb{R}^{m\times n}$, $m < n$, is a matrix of full rank. The linear system given in (1) is nonsingular and has a unique solution, if the block $\mathcal{A}$ is positive definite on the kernel of B [5].

All at once methods are applied mostly to OCP problems governed by linear elliptic or parabolic partial differential equations (PDEs) [6,7] and the Stokes equation [8]. Recently optimal control problems with coupled nonlinear diffusion–reaction equations are solved by the all-at-once approach [9]. Standard time integrators for the Burgers equation are the Crank–Nicolson and backward Euler methods which are implicit and unconditionally stable. The Burgers equation can also be solved by the semi-implicit method, which provides an effective linearization procedure by solving, at each time step, a linear system of equations with the same symmetric matrix [10–12]. In the semi-implicit method, the diffusive part of the Burgers equation is discretized implicitly, and the non-linear part explicitly. It is also unconditionally stable as the backward Euler and Crank–Nicolson methods. It is first order convergent in time for the linear diffusion–convection equations [12] and for the Burgers equation like the backward Euler, whereas the Crank–Nicolson method is of second order. Using piecewise linear finite-element in space and semi-implicit discretization in time, the linearized OCP problem is solved sequentially at each time step. In practice, the linear system $\mathcal{A}x = b$ is usually of large size. When a one-shot approach for time-dependent problems in the two dimensional space is considered, iterative solution methods with preconditioners are needed [13,14].

There are two different approaches for the discretization of the OCPs: *optimize then discretize* (OD) and *discretize then optimize* (DO). In the OD approach, first the necessary optimality conditions are established on the continuous level consisting of the state, adjoint and the optimality equations, and then these equations are usually discretized by finite elements. In the DO approach, the state equation is discretized and then the optimality system for the finite dimensional optimization problem is derived. Recently, the commutativity of DO and OD approaches is discussed for solving OPC problems with linear PDE constraints (see [15] for an overview). We will show that using the DO, the first order optimality conditions for the linearized Burgers equation lead to a symmetric saddle point problem, and the OD to a non-symmetric linear system. We follow here the DO, because the OD approach implies that there is no finite dimensional optimization problem.

The paper is organized as follows. The discretization of the problem by finite elements in space and the time discretization with the semi-implicit method will be presented in Section 2. Also the application of the all-at-once method for the distributed unconstrained and control constrained problems is given in Section 2. Numerical results are given in Section 3.

2. The distributed control problem and discretization

We define $\Omega = (0, 1)$, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$ for given $T > 0$. The distributed control problem for the viscous Burgers equation with control constraints and with homogeneous Dirichlet boundary conditions can be stated as follows [16,1]:

$$\min_{(y,u)} J(y, u) = \frac{1}{2}\int_0^T \int_0^1 \left((y(x, t) - y_d(x, t))^2 + \alpha u^2(x, t)\right) dxdt, \tag{2}$$

$$\begin{aligned} &\text{subject to } y_t + yy_x - \nu y_{xx} = f + u \quad (x, t) \in Q, \\ &y(0, t) = y(1, t) = 0 \quad t \in \Sigma, \\ &y(x, 0) = y_0 \quad x \in \Omega, \end{aligned} \tag{3}$$

with bound constraints on the control

$$u_a(t, x) \le u(t, x) \le u_b(t, x) \quad \text{a.e. in } Q,$$

where u_a and u_b are given functions in $Ł^\infty(Q)$ satisfying $u_a \le u_b$. Here $y(x, t)$ and $y_d(x, t) \in L^2(Q)$ denote the state and the desired state, respectively. $\nu > 0$ is the viscosity and $\alpha > 0$ is the regularization parameter. For a fixed function $f \in L^2(Q)$, a given initial condition $y_0 \in H_0^1(\Omega)$ and for the control $u \in L^2(Q)$, the existence and uniqueness conditions of the weak solution of the Burgers equation are given in [1].

Discretize-then-optimize approach.

In the DO approach, first the state equation and the objective function are discretized and the optimality conditions are derived by using the discrete Lagrangian. State and control variables are discretized by using the standard Galerkin method with linear finite elements in space with n uniform subdivisions with the step size $h = 1/n$. The weak form of the Burgers equation becomes

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \left(\sum_{j=1}^{n-1} y_j\phi_j\right)\phi_i dx + \nu\int_0^1 \frac{d}{dx}\left(\sum_{j=1}^{n-1} y_j\phi_j\right)\frac{d}{dx}\phi_i dx + \int_0^1 \frac{d}{dx}\left(\sum_{k=1}^{n-1} y_k\phi_k\right)\left(\sum_{j=1}^{n-1} y_j\phi_j\right)\phi_i dx \\ &\quad = \int_0^1 f(x, t)\phi_i(x)dx + \int_0^1 \left(\sum_{j=1}^{n-1} u_j\phi_j\right)\phi_i dx, \quad i = 1, \ldots, n-1. \end{aligned}$$