Journal of Computational and Applied Mathematics 259 (2014) 790–797

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Solution of initial and boundary value problems by the variational iteration method

D. Altıntan [a], Ö. Uğur [b,*]

[a] Department of Mathematics, Selçuk University, 42697 Konya, Turkey

[b] Institute of Applied Mathematics, Middle East Technical University, 06800 Ankara, Turkey

ARTICLE INFO

*Article history:*
Received 15 February 2013
Received in revised form 1 June 2013

*Keywords:*
Variational iteration method
Initial value problems
Boundary value problems
Green's function

ABSTRACT

The Variational Iteration Method (VIM) is an iterative method that obtains the approximate solution of differential equations. In this paper, it is proven that whenever the initial approximation satisfies the initial conditions, VIM obtains the solution of Initial Value Problems (IVPs) with a single iteration. By using this fact, we propose a new algorithm for Boundary Value Problems (BVPs): linear and nonlinear ones. Main advantage of the present method is that it does not use Green's function, however, it has the same effect that it produces the exact solution to linear problems within a single, but simpler, integral. In order to show the effectiveness of the method we give some examples including linear and nonlinear BVPs.



## 1. Introduction

The Variational Iteration Method (VIM) which was proposed by He in 1997 is an iterative method that approximates solutions of differential equations [1–3]. The method's close relation with other well-known methods, such as Picard–Lindelöf and fixed-point iterations, were pointed in [4,5]. Over the years, the VIM has been applied to various types of problems from different areas and it has been compared to many other methods. See, for instance, [6–12].

The method is a modification of the general Lagrange multiplier [13]: the VIM constructs a correctional functional which uses an initial function to obtain a better approximate solution. The key element of the correction functional is the so-called Lagrange multiplier which can be identified via *variational theory*.

In this paper, we analyse some basic properties of the Lagrange multiplier and by using these properties we propose a new algorithm for initial and boundary value problems. In [6], a new approach of VIM for first order differential equations has been proposed. In this approach matrix valued Lagrange multipliers were defined and the close relation between the Lagrange multipliers and fundamental matrices of the homogeneous systems was shown. This leads us to prove that for higher order differential equations when the initial approximation satisfies the initial condition, the solution of the initial value problem can be obtained with a single variational iteration.

Consequently, it is shown that if the initial approximation satisfies the boundary condition of linear boundary value problems then it is possible to obtain the solution of boundary value problems with a single step, too. The main advantage of the proposed algorithm is that it does not use any theory of Green's function. Moreover, we extend this algorithm to nonlinear boundary value problems and illustrate it with numerical examples.

The organisation of the paper is as follows: in Section 2, we summarise some of the basic properties of matrix-valued and scalar-valued Lagrange multipliers to show that the VIM is a very effective method for initial value problems. In Section 3, we

* Corresponding author. Tel.: +90 3122105617; fax: +90 3122102985.
*E-mail addresses:* altintan@selcuk.edu.tr (D. Altıntan), ougur@metu.edu.tr (Ö. Uğur).

0377-0427/$ – see front matter 
http://dx.doi.org/10.1016/j.cam.2013.07.012

show that similar results for initial value problems are also valid for boundary value problems, especially, for nonlinear ones. Hence, we propose a new algorithm for boundary value problems. We illustrate the results obtained with some examples. Finally, in Section 4 we present a brief summary.

## 2. Solution of initial value problems

In this section of the present work, following the results of [6] we will analyse some basic properties of the Lagrange multiplier for the VIM. Moreover, by using these properties we will prove that whenever the initial approximation satisfies the initial condition, the solution of $m$th-order linear homogeneous ordinary differential equations can be obtained by only a single step of VIM.

Consider the following $m$th-order linear homogeneous differential equation

$$p_0(t)\dot{x}^{(m)}(t) + p_1(t)\dot{x}^{(m-1)}(t) + \cdots + p_m(t)x(t) = 0, \tag{1}$$

subject to the initial conditions

$$x(t_0) = \alpha_1, \qquad \dot{x}(t_0) = \alpha_2, \ldots, \dot{x}^{(m-1)}(t_0) = \alpha_m,$$

where $t \in I = [a, b]$, and $t_0 \in I$, $p_i \in C^{m-i}(I, \mathbb{R})$, $p_0(t) > 0$ for all $t \in I$, and $\dot{x}^{(i)}$ represents the $i$th derivative $d^i x(t)/dt^i$ for all $i = 1, 2, \ldots, m$.

The VIM gives the correction functional for (1) as

$$x_{n+1}(t) = x_n(t) + \int_{t_0}^{t} \lambda(s; t) L_{m,s} x_n(s) ds \tag{2}$$

where $L_{m,s}$ is the following linear differential operator

$$L_{m,s} = p_0(s)\frac{d^m}{ds^m} + p_1(s)\frac{d^{m-1}}{ds^{m-1}} + \cdots + p_m(s). \tag{3}$$

By using the calculus of variations, integration by parts, and imposing the conditions $\delta x_n(t_0) = 0$, we found out that, at $s = t$, the conditions

$$0 = \{(-1)^{m-1}(\lambda(s;t)p_0(s))^{(m-1)} + (-1)^{m-2}(\lambda(s;t)p_1(s))^{(m-2)} + \cdots + \lambda(s;t)p_{m-1}(s) + 1\}|_{s=t}, \tag{4a}$$

$$0 = \{(-1)^{m-2}(\lambda(s;t)p_0(s))^{(m-2)} + (-1)^{m-3}(\lambda(s;t)p_1(s))^{(m-3)} + \cdots + \lambda(s;t)p_{m-2}(s)\}|_{s=t}, \tag{4b}$$

$$\vdots$$

$$0 = p_0(s)\lambda(s;t)|_{s=t}, \tag{4c}$$

must be fulfilled so that

$$L^{\dagger}_{m,s}\lambda(s;t) = 0, \tag{5}$$

where the adjoint operator $L^{\dagger}_{m,s}$ is

$$L^{\dagger}_{m,s}(\cdot) = (-1)^m \frac{d^m}{ds^m}(p_0(s)\cdot) + (-1)^{m-1}\frac{d^{m-1}}{ds^{m-1}}(p_1(s)\cdot) + \cdots + (p_m(s)\cdot).$$

Using the fact that $p_0(t) > 0$, (4c) gives $\lambda(t;t) = 0$; backward substitution in (4) reveals that the derivatives of the Lagrange multiplier should satisfy

$$\begin{aligned} \left.\frac{\partial^j \lambda(s;t)}{\partial s^j}\right|_{s=t} &= 0, \quad \text{for } j = 0, 1, 2, \ldots, m-1, \\ \left.\frac{\partial^{m-1}\lambda(s;t)}{\partial s^{m-1}}\right|_{s=t} &= \frac{(-1)^m}{p_0(t)}. \end{aligned} \tag{6}$$

On the other hand, the associated system of equations for (1) can be written in matrix–vector form as

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \tag{7}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T = (x, \dot{x}, \ldots, \dot{x}^{(m-1)})^T$ and $A(t)$ is the so-called *companion* matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -\tilde{p}_m & -\tilde{p}_{m-1} & -\tilde{p}_{m-2} & \cdots & -\tilde{p}_1 \end{pmatrix} \tag{8}$$