Journal of Computational and Applied Mathematics 259 (2014) 815–827

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Numeric and mesh algorithms for the Coxeter spectral study of positive edge-bipartite graphs and their isotropy groups

Rafał Bocian, Mariusz Felisiak, Daniel Simson *

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland

ARTICLE INFO

Article history:
Received 13 February 2013
Received in revised form 30 April 2013

Keywords:
Edge-bipartite graph
Matrix morsification
Dynkin diagram
Coxeter polynomial
Mesh geometry of roots
Computer algorithm

ABSTRACT

We develop algorithmic techniques for the Coxeter spectral analysis of the class $\mathcal{UB}igr_n$ of connected loop-free positive edge-bipartite graphs Δ with $n \geq 2$ vertices (i.e., signed graphs). In particular, we present numerical and graphical algorithms allowing us a computer search in the study of such graphs Δ by means of their Gram matrix $\check{G}_\Delta$, the (complex) spectrum $\mathbf{specc}_\Delta \subseteq \mathbb{C}$ of the Coxeter matrix $\mathrm{Cox}_\Delta := -\check{G}_\Delta \cdot \check{G}_\Delta^{-tr}$, and the geometry of Weyl orbits in the set $\mathbf{Mor}_{D\Delta}$ of matrix morsifications $A \in \mathbb{M}_n(\mathbb{Z})$ of a simply laced Dynkin diagram $D\Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ associated with Δ and mesh root systems of type $D\Delta$. Our algorithms construct the Coxeter–Gram polynomials $\mathrm{cox}_\Delta(t) \in \mathbb{Z}[t]$ and mesh geometries of root orbits of small connected loop-free positive edge-bipartite graphs Δ. We apply them to the study of the following Coxeter spectral analysis problem: *Does the* $\mathbb{Z}$*-congruence* $\Delta \approx_\mathbb{Z} \Delta'$ *hold (i.e., the matrices* $\check{G}_\Delta$ *and* $\check{G}_{\Delta'}$ *are* $\mathbb{Z}$*-congruent), for any pair of connected positive loop-free edge-bipartite graphs* Δ, Δ' *in* $\mathcal{UB}igr_n$ *such that* $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$? The problem if any square integer matrix $A \in \mathbb{M}_n(\mathbb{Z})$ is $\mathbb{Z}$-congruent with its transpose A^{tr} is also discussed. We present a solution for graphs in $\mathcal{UB}igr_n$, with $n \leq 6$.

© 2013 Elsevier B.V. All rights reserved.

1. Preliminaries and notation

One of the aims of the paper is to develop algorithmic techniques for the study of Coxeter spectral analysis problems formulated in [1–3] for loop-free edge-bipartite graphs $\Delta = (\Delta_0, \Delta_1)$, with $n \geq 2$ vertices. Here we keep the terminology and notation introduced in [3,4]. In particular, we denote by $\mathbb{N}$ the set of non-negative integers, by $\mathbb{Z}$ the ring of integers, and by $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ the rational, the real, and the complex number field, respectively. We view $\mathbb{Z}^n$, with $n \geq 1$, as a free abelian group, and we denote by $e_1, \ldots, e_n$ the standard $\mathbb{Z}$-basis of $\mathbb{Z}^n$. We denote by $\mathbb{M}_n(\mathbb{Z})$ the $\mathbb{Z}$-algebra of all square n by n matrices, by $E \in \mathbb{M}_n(\mathbb{Z})$ the identity matrix. We also use the right action $* : \mathbb{M}_n(\mathbb{Q}) \times \mathrm{Gl}(n, \mathbb{Z}) \longrightarrow \mathbb{M}_n(\mathbb{Q})$, $(A, C) \mapsto A * C := C^{tr} \cdot A \cdot C$ of the general $\mathbb{Z}$-linear group $\mathrm{Gl}(n, \mathbb{Z}) := \{A \in \mathbb{M}_n(n, \mathbb{Z}),\ \det A \in \{-1, 1\}\}$ on $\mathbb{M}_n(\mathbb{Q})$.

Following [3,4], by an *edge-bipartite graph* (bigraph, for short), we mean a pair $\Delta = (\Delta_0, \Delta_1)$, where Δ_0 is a finite non-empty set of vertices and Δ_1 is a finite set of edges equipped with a *bipartition* $\Delta_1 = \Delta_1^- \cup \Delta_1^+$ such that the set $\Delta_1(i,j) = \Delta_1^-(i,j) \cup \Delta_1^+(i,j)$ of edges connecting the vertices i and j does not contain edges lying in $\Delta_1^-(i,j) \cap \Delta_1^+(i,j)$, for each pair of vertices $i, j \in \Delta_0$, and either $\Delta_1(i,j) = \Delta_1^-(i,j)$ or $\Delta_1(i,j) = \Delta_1^+(i,j)$. Obviously, edge-bipartite graphs can be viewed as signed multi-graphs satisfying a separation property, see [3,5]. We call $\Delta = (\Delta_0, \Delta_1)$ *loop-free* if it has no loops, that is, $\Delta_1(j,j) = \emptyset$, for all $j \in \Delta_0$. We denote by $\mathcal{B}igr_n$ the category of finite edge-bipartite graphs, with $n \geq 2$ vertices, and the usual edge-bipartite graph maps as morphisms, see [3] for details. The full subcategory of $\mathcal{B}igr_n$ consisting of all loop-free graphs is denoted by $\mathcal{UB}igr_n$.

* Corresponding author.
E-mail addresses: rafalb@mat.uni.torun.pl (R. Bocian), felixx@mat.uni.torun.pl (M. Felisiak), simson@mat.umk.pl, simson@mat.uni.torun.pl (D. Simson).

0377-0427/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cam.2013.07.013

We visualize Δ as a graph in a Euclidean space $\mathbb{R}^m$, $m \geq 2$, with the vertices $a_1, \dots, a_n$ numbered by the integers $1, \dots, n$; usually we simply write $\Delta_0 = \{1, \dots, n\}$. An edge in $\Delta_1^-(a_i, a_j)$ is visualized as a continuous one a_i——— a_j, and an edge in $\Delta_1^+(a_i, a_j)$ is visualized as a dotted one a_i- - -a_j.

We view any finite graph $\Delta = (\Delta_0, \Delta_1)$ as an edge-bipartite one by setting $\Delta_1^-(a_i, a_j) = \Delta_1(a_i, a_j)$ and $\Delta_1^+(a_i, a_j) = \emptyset$, for each pair of vertices $a_i, a_j \in \Delta_0$. We study the loop-free edge-bipartite graphs $\Delta \in \mathcal{UB}igr_n$ by means of the *non-symmetric Gram matrix*

$$\check{G}_\Delta = [d_{ij}^\Delta] \in \mathbb{M}_n(\mathbb{Z}),$$

where $d_{ij}^\Delta = -|\Delta_1^-(a_i, a_j)|$, if there is an edge a_i——— a_j and $i \leq j$, $d_{ij}^\Delta = |\Delta_1^+(a_i, a_j)|$, if there is an edge a_i- - -a_j and $i \leq j$. We set $d_{ij}^\Delta = 0$, if $\Delta_1(a_i, a_j)$ is empty or $j < i$. The matrix $G_\Delta := \frac{1}{2}(\check{G}_\Delta + \check{G}_\Delta^{tr})$ is called the *symmetric Gram matrix*. We call $\Delta = (\Delta_0, \Delta_1)$ *positive* (resp. non-negative), if the rational symmetric Gram matrix $G_\Delta := \frac{1}{2}(\check{G}_\Delta + \check{G}_\Delta^{tr})$ of Δ is positive definite (resp. positive semi-definite). Two graphs $\Delta, \Delta' \in \mathcal{B}igr_n$ are defined to be $\mathbb{Z}$-*equivalent* (resp. $\mathbb{Z}$-*bilinear equivalent*) if there exists $B \in \mathrm{Gl}(n, \mathbb{Z})$ such that $G_{\Delta'} = B^{tr} \cdot G_\Delta \cdot B$ (resp. $\check{G}_{\Delta'} = B^{tr} \cdot \check{G}_\Delta \cdot B$). In this case, we write $\Delta \sim_\mathbb{Z} \Delta'$ (resp. $\Delta \approx_\mathbb{Z} \Delta'$) and we say that $B \in \mathrm{Gl}(n, \mathbb{Z})$ defines the $\mathbb{Z}$-equivalence $\Delta \sim_\mathbb{Z} \Delta'$ (resp. $\Delta \approx_\mathbb{Z} \Delta'$).

Following [3] (see also [6]), we associate with any edge-bipartite graph Δ in $\mathcal{UB}igr_n$, with $n \geq 2$, the *Coxeter spectrum* $\mathbf{specc}_\Delta \subseteq \mathbb{C}$, i.e., the spectrum of the *Coxeter(–Gram) matrix* and of the *Coxeter(–Gram) polynomial*

$$\mathrm{Cox}_\Delta := -\check{G}_\Delta \cdot \check{G}_\Delta^{-tr} \in \mathbb{M}_n(\mathbb{Z}), \qquad \mathrm{cox}_\Delta(t) := \det(t \cdot E - \mathrm{Cox}_\Delta) \in \mathbb{Z}[t]. \tag{1.1}$$

The Coxeter transformation $\Phi_\Delta : \mathbb{Z}^n \to \mathbb{Z}^n$ of Δ is defined by $\Phi_\Delta(v) := v \cdot \mathrm{Cox}_\Delta$ and the Coxeter number $\mathbf{c}_\Delta$ of Δ is a minimal integer $c \geq 2$ such that Φ_Δ^c is the identity map on $\mathbb{Z}^n$. By the *integral quadratic form* of Δ we mean

$$q_\Delta(x) := b_\Delta(x, x) = x_1^2 + \cdots + x_n^2 + \sum_{i<j} d_{ij}^\Delta x_i x_j = x \cdot G_\Delta \cdot x^{tr} = x \cdot \check{G}_\Delta \cdot x^{tr}. \tag{1.2}$$

We recall from [3, Lemma 2.1] that the Coxeter spectrum $\mathbf{specc}_\Delta \subseteq \mathbb{C}$ lies on the unit circle $\mathcal{S}^1 := \{z \in \mathbb{C};\ |z| = 1\}$ and all points in $\mathbf{specc}_\Delta$ are roots of unity, if Δ is non-negative. If, in addition, Δ is positive then $1 \notin \mathbf{specc}_\Delta$ and the set $\mathcal{R}_\Delta := \{v \in \mathbb{Z}^n;\ v \cdot G_\Delta \cdot v^{tr} = 1\} \subseteq \mathbb{Z}^n$ of *roots of* Δ is finite.

One of our aims of the paper is to present an algorithmic technique for a computer search of the following Coxeter spectral analysis problems stated in [3] and discussed in [2–4,7].

Problem 1.3. Given $n \geq 2$, compute the set $\mathcal{CG}pol_n^+$ of all Coxeter(–Gram) polynomials $\mathrm{cox}_\Delta(t) \in \mathbb{Z}[t]$, with positive connected loop-free edge-bipartite graphs Δ in $\mathcal{UB}igr_n$.

Problem 1.4. Show that, given a pair of connected positive loop-free edge-bipartite graphs Δ and Δ' in $\mathcal{UB}igr_n$, the equality $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$ is equivalent to the existence of a $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_n(\mathbb{Z})$ such that $\check{G}_{\Delta'} = B^{tr} \cdot \check{G}_\Delta \cdot B$. Construct an algorithm computing such a matrix $B \in \mathrm{Gl}(n, \mathbb{Z})$.

Problem 1.5. For any matrix $A \in \mathbb{M}_n(\mathbb{Z})$, with $\det A = 1$, find a matrix $C \in \mathrm{Gl}(n, \mathbb{Z})$ such that $A^{tr} = C^{tr} \cdot A \cdot C$ and $C^2 = E$, see [8].

Problem 1.6. Given a connected positive loop-free edge-bipartite graph Δ in $\mathcal{UB}igr_n$, construct a minimal Φ_Δ-mesh geometry of roots of Δ (that is, a Φ_Δ-mesh translation quiver $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$ satisfying the conditions of [2, Definition 1.11], see Section 2) such that, for any a pair of connected positive edge-bipartite graphs Δ and Δ' in $\mathcal{UB}igr_n$, the equality $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$ implies the existence of a group automorphism $\mathbb{Z}^n \cong \mathbb{Z}^n$ that restricts to a Φ_Δ-mesh translation quiver isomorphism $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta) \cong \Gamma(\widehat{\mathcal{R}}_{\Delta'}, \Phi_{\Delta'})$.

The main results of the paper are the following two theorems (proved in Section 3) that contain a partial solution of Problems 1.3–1.6 for edge-bipartite graphs Δ, Δ' in $\mathcal{UB}igr_n$, with $n \leq 6$.

Theorem 1.7. *Assume that Δ, Δ' are positive connected loop-free edge-bipartite graphs in $\mathcal{UB}igr_n$, with $2 \leq n \leq 6$ and $D\Delta, D\Delta'$ are the simply laced Dynkin diagrams associated in Theorem 2.1, with $\Delta \sim_\mathbb{Z} D\Delta$ and $\Delta' \sim_\mathbb{Z} D\Delta'$.*

(a) $(\mathrm{cox}_\Delta(t), \mathbf{c}_\Delta)$ *is one of the pairs* $(F_{D\Delta}^{(j)}(t), \mathbf{c}_{D\Delta}^{(j)})$ *listed in Table 1.8, see also [3, Figure 3].*

(b) $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$ *if and only if* $\Delta \approx_\mathbb{Z} \Delta'$.

(c) *Given Δ, there exists a matrix $C \in \mathrm{Gl}(n, \mathbb{Z})$ such that $\check{G}_\Delta^{tr} = C^{tr} \cdot \check{G}_\Delta \cdot C$ and $C^2 = E$.*

(d) *Given Δ, there is a minimal Φ_Δ-mesh geometry of roots $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$ of Δ satisfying the conditions listed in 1.6.*

The following theorem contains a complete classification of positive connected loop-free edge-bipartite graphs with at most six vertices, up to the congruence $\Delta \approx_\mathbb{Z} \Delta'$.

Theorem 1.9. *Assume that Δ is a positive connected loop-free edge-bipartite graph in $\mathcal{UB}igr_n$, with $2 \leq n \leq 6$. Under the notation in Theorem 1.7, we have*

(a) *If* $\mathrm{cox}_\Delta(t) = F_{D\Delta}^{(1)}(t)$ *then* $\Delta \approx_\mathbb{Z} D\Delta$.

Download English Version:

https://daneshyari.com/en/article/6422755

Download Persian Version:

https://daneshyari.com/article/6422755

Daneshyari.com