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Estimation of the Hurst parameter for fractional Brownian motion using the CMARS method

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ABSTRACT

In this study, we develop an alternative method for estimating the Hurst parameter using the conic multivariate adaptive regression splines (CMARS) method. We concentrate on the strong solutions of stochastic differential equations (SDEs) driven by fractional Brownian motion (fBm). Our approach is superior to others in that it not only estimates the Hurst parameter but also finds spline parameters of the stochastic process in an adaptive way. We examine the performance of our estimations using simulated test data.

1. Introduction

Fractional Brownian motion (fBm) is a widely used concept in modeling various features such as the level of water in a river, the temperature at a specific place, the empirical volatility of a stock, and the price dynamics of electricity. It appears naturally in these phenomena because of its capability of explaining the *dependence* structure in real-life observations. A main purpose of introducing the concept of an fBm, $W^H(\cdot)$, lies in a notion of random fluctuation of a time-continuous stochastic processes, $X(\cdot)$, which is wider than that given by a Brownian motion $W(\cdot)$. In fact, the connection between $X(\cdot)$ and $W(\cdot)$ or, in our case, $W^H(\cdot)$, is implied by our model which is a stochastic differential equation (SDE), where the (fractional) Brownian motion is a key component in the second, actually, random or *diffusion* term. Since the (fractional) Brownian motion fulfills certain axioms, it can be regarded as a formatted or normalized random fluctuation; moreover, fBm is a continuous zero-mean Gaussian process with stationary increments. Therefore, in the SDE a factor occurs in front of the differential (fractional) Brownian term; that factor plays the role of volatility. The fBm is characterized by a parameter, the so-called Hurst parameter H. An fBm with Hurst parameter $H > 1/2$ is called a *persistent* process, i.e., the increments of this process are positively correlated. On the other hand, the increments of an fBm with $H < 1/2$ constitute what is called an *anti-persistent* process, with increments being negatively correlated. For $H = 1/2$, an fBm corresponds to Brownian motion which has independent increments. For further information on fBm and its applications, see [1–3]. We will estimate

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the Hurst parameter together with the model coefficients that will be linearly involved in the representation of the entire SDE—to be more precise, in the time-discrete approximation which we shall study. This estimation will be based on given data and supported by modern optimization techniques. In our paper, the aforementioned items will become represented more closely.

It is highly important to identify the value of the Hurst parameter in order to understand the structure of the process and its applications, since the calculations differ dramatically according to the value of H. Therefore, some techniques have been developed for estimating the Hurst parameter which can be categorized into three groups; heuristics, maximum likelihood and wavelet-based estimators. In the group of heuristics estimators, there is the R/S estimator which was first proposed by Hurst [4], followed by the methods based on correlograms, variograms, variance plots, and partial correlations plots. Due to the lack of accuracy of heuristics estimators, maximum likelihood estimators (mle) were developed. Being weakly consistent is the main disadvantage of mle. In parallel with mle, wavelet-based estimators were suggested, because of the popularity of wavelet decomposition of fBm [5,6].

In search of faster and more efficient ways to estimate the Hurst parameter H, we suggest a new numerical and computational approach: using *conic multivariate adaptive regression splines (CMARS)*. Using the CMARS approach is an alternative to using the well-known data mining tool *multivariate adaptive regression splines (MARS)*. It is based on a penalized residual sum of squares (PRSS) for MARS as a Tikhonov regularization (TR) problem. CMARS treats this problem by a continuous optimization technique, in particular using the framework of *conic quadratic programming (CQP)*. These convex optimization problems are globally very well-structured, thereby resembling linear programs and, hence, permitting the use of *interior point methods*.

This paper is organized as follows. In Section 2, we start with explaining the properties of our model given as SDEs driven by fBm. In Section 3, we introduce the CMARS method, relating it to the Hurst parameter estimation of our model. In Section 4, we give an application of our study, in order to test the theory that we have developed. Finally, we present a brief conclusion and a general outlook of our study.

2. Stochastic differential equations with fractional Brownian motion

SDEs generated by fBm are widely used to represent noisy and real-world problems. They play an important role in many fields of science such as finance, physics, biotechnology and engineering. In this section, we briefly recall some concepts relating to fBm and stochastic differential equations driven by fBm.

2.1. Fractional Brownian motion

Let H be a constant in the interval $(0, 1)$. The fBm $(W^H(t))_{t\geq 0}$ with Hurst parameter H is a continuous and centered Gaussian process with covariance function

$$E[W^H(t)W^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

We note that, for $H = 1/2$, the fBm corresponds to a standard Brownian motion which has independent increments. For a *standard fBm*, $W^H(t)$:

- $W^H(0) = 0$ and $E[W^H(t)] = 0$ for all $t \geq 0$.
- W^H has homogeneous increments, i.e., $W^H(t+s) - W^H(s)$ has the same law as $W^H(t)$, for all $s, t \geq 0$.
- W^H is a Gaussian process and $E[(W^H(t))^2] = t^{2H}$ $(t \geq 0)$, for all $H \in (0, 1)$.
- W^H has continuous trajectories.

The Hurst parameter H of the fBm explains the dependence of data [1,7,3]. Indeed, the correlation between increments for $s, t \geq 0$ can be obtained by using

$$\mathbb{E}[(W^H(t+h) - W^H(t))(W^H(s+h) - W^H(s))] = \frac{h^{2H}}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}].$$

It can be seen that observations with $H > 1/2$ have positively correlated increments and display long-range dependence, while the observations with $H < 1/2$ have negatively correlated increments and display a short-range dependence structure (see Fig. 1). Therefore, it is crucial to find the Hurst parameter of a stochastic process for understanding many phenomena in diverse fields from engineering to finance. For example, it is observed that the prices of electricity in a liberated electricity market have spikes which can be regarded as negatively correlated increments. This phenomenon can be modeled by an SDE driven by an fBm with $H < 1/2$. On the other hand, in financial markets, the prices of stocks usually display long-range dependence which can be explained by an SDE driven by an fBm with $H > 1/2$. In this study, we concentrate on finding H for the stochastic processes which are the strong solutions of SDEs with fBm. Hence, we first recall some fundamental properties of them.