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Criterions for the positive definiteness of real supersymmetric tensors

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ABSTRACT

The positive definiteness of an even-degree homogeneous polynomial form $f(x)$ plays an important role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control, and the positive definiteness of $f(x)$ is equivalent to that of an even-order supersymmetric tensor which defines $f(x)$. In this paper, we provide some criterions for identifying the positive definiteness of an even-order real supersymmetric tensor. Moreover, an iterative algorithm for identifying the positive definiteness of an even-order real supersymmetric tensor is obtained. Numerical examples are given to verify the corresponding results.

1. Introduction

A complex (real) order m dimension n tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ consists of n^m complex (real) entries:

$$a_{i_1\cdots i_m} \in \mathbb{C}(\mathbb{R}),$$

where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$ [1–5]. It is obvious that a matrix is an order 2 tensor. Moreover, a tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is called supersymmetric [6,7] if

$$a_{i_1\cdots i_m} = a_{\pi(i_1\cdots i_m)}, \quad \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. An order m dimension n tensor is called the unit tensor [8], if its entries are $\delta_{i_1\cdots i_m}$ for $i_1, \ldots, i_m \in N = \{1, 2, \ldots, n\}$, where

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Given an order m dimension n complex tensor $\mathcal{A} = (a_{i_1\cdots i_m})$, if there are a complex number λ and a nonzero complex vector $x = (x_1, x_2, \ldots, x_n)^T$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

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then λ is called the eigenvalue of $\mathcal{A}$ and x the eigenvector of $\mathcal{A}$ associated with λ [9–12,7,13–16], where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose ith components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$(x^{[m-1]})_i = x_i^{m-1},$$

respectively. And if λ, x and every entry of $\mathcal{A}$ are restricted in the real field, then λ is called the H-eigenvalue of $\mathcal{A}$ and x an H-eigenvector of $\mathcal{A}$ associated with λ [17,7,18]. Qi [17,7] also called a real number λ and a real vector $x \in \mathbb{R}^n$ a Z-eigenvalue of $\mathcal{A}$ and a Z-eigenvector of $\mathcal{A}$ associated with the Z-eigenvalue λ respectively, if they are solutions of the following system:

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^T x = 1. \end{cases}$$

Consider the following positive definiteness identification problem [19].

Problem 1. For an mth degree homogeneous polynomial of n variables $f(x)$ denoted as

$$f(x) = \sum_{i_1,\ldots,i_m \in N} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}, \quad (1)$$

where $x \in \mathbb{R}^n$, how to check whether $f(x)$ is positive definite, i.e.,

$$f(x) > 0 \quad \text{for any } x \in \mathbb{R}^n,\ x \neq 0,$$

or not?

Problem 1 appears in numerous application domains; see [20–31]. The positive definiteness of multivariate polynomial $f(x)$ plays an important role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control [20,22–26,32,27,29,31], such as the multivariate network realizability theory [24], a test for Lyapunov stability in multivariate filters [22], a test of existence of periodic oscillations using Bendixon's theorem [32], and the output feedback stabilization problems [20].

The homogeneous polynomial $f(x)$ in (1) is equivalent to the tensor product of an order m dimensional n supersymmetric tensor $\mathcal{A}$ and x^m defined by

$$f(x) = \mathcal{A}x^m = \sum_{i_1,\ldots,i_m \in N} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}, \quad (2)$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$; see [28]. For $n \leq 3$, the positive definiteness of the homogeneous polynomial form (2) can be checked by a method based on the Sturm theorem [23]. For $n > 3$ and $m \geq 4$, the methods in [22,23,28] cannot be used practically for testing the positive definiteness of such a multivariate form; see [3]. In [7], Qi pointed out that $f(x)$ defined by (2) is positive definite if and only if the real supersymmetric tensor $\mathcal{A}$ is positive definite, and provided an eigenvalue method to verify the positive definiteness of $\mathcal{A}$ when m is even (see Theorem 1).

Theorem 1 *([7, Theorem 5]).* *Let $\mathcal{A}$ be an even-order real supersymmetric tensor. Then*

(1) *$\mathcal{A}$ is positive definite if and only if all of its H-eigenvalues are positive;*
(2) *$\mathcal{A}$ is positive definite if and only if all of its Z-eigenvalues are positive.*

Remark here that from Theorem 1, we can identify whether an even-order real supersymmetric tensor $\mathcal{A}$ is positive definite by using all H-eigenvalues or Z-eigenvalues of $\mathcal{A}$. Based on the Gershgorin-type theorem [7, Theorem 6] for H-eigenvalues (also see Lemma 3), we can obtain some inequalities to identify the positive definiteness of $\mathcal{A}$. However, the Gershgorin-type theorem does not hold for Z-eigenvalues (also see page 1305 of [7]). Hence, we only use H-eigenvalues to identify the positive definiteness of $\mathcal{A}$.

From Theorem 1, we can verify the positive definiteness of an even-order supersymmetric tensor $\mathcal{A}$ (the positive definiteness of the mth-degree homogeneous polynomial $f(x)$ defined by (2)) by computing the H-eigenvalues of $\mathcal{A}$. However, it is difficult to compute all the H-eigenvalues or the smallest H-eigenvalue of an order m dimension n real tensor $\mathcal{A}$ when m and n are large. Therefore, finding effective methods to identify the positive definiteness of a tensor is interesting [25,26,33,17,19,31].

In this paper, we research the positive definiteness identification problem. In Section 2, some criterions for identifying the positive definiteness of an even-order real supersymmetric tensor are obtained. In Section 3, we give an iterative algorithm for identifying the positive definiteness of an even-order real supersymmetric tensor based on the results obtained in Section 2. Numerical examples are also given to verify the corresponding results.

Now some definitions and notation are given, which will be used in the sequel. Vectors are written as italic lowercase letters such as, $x, y, \ldots,$ matrices correspond to italic capitals such as, $A, B, \ldots,$ and tensors are written as calligraphic