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Complexity of interior-point methods for linear optimization based on a new trigonometric kernel function

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ABSTRACT

In this paper, we propose a new kernel function with trigonometric barrier term for primal–dual interior point methods in linear optimization. Using an elegant and simple analysis and under some easy to check conditions, we explore the worst case complexity result for the large update primal–dual interior point methods. We obtain the worst case iteration bound for the large update primal–dual interior point methods as $O\left(n^{\frac{2}{3}}\log\frac{n}{\epsilon}\right)$ which improves the so far obtained complexity results for the trigonometric kernel function in [M. El Ghami, Z.A. Guennoun, S. Boula, T. Steihaug, Interior-point methods for linear optimization based on a kernel function with a trigonometric barrier term, Journal of Computational and Applied Mathematics 236 (2012) 3613–3623] significantly.

1. Introduction

After the landmark paper of Karmarkar [1], the interior point methods became an active area of research. Nowadays, introducing an efficient variant of the polynomial time interior point methods with low complexity results is the main challenge in this area of research.

In this paper, we deal with primal–dual Interior-Point Methods (IPMs) for solving the standard Linear Optimization (LO) problem:

$$\min\{c^T x : Ax = b, x \geq 0\} \tag{P}$$

where $A \in \mathbb{R}^{m\times n}$ with $rank(A) = m \leq n, x, c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The dual problem of (P) is given by

$$\max\{b^T y : A^T y + s = c, s \geq 0\} \tag{D}$$

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$. One knows that the necessary and sufficient conditions for the existence of optimal solutions of (P) and (D) lead us to the following nonlinear system:

$$\begin{aligned} &Ax = b, \quad x \geq 0\\ &A^T y + s = c, \quad s \geq 0\\ &xs = 0. \end{aligned} \tag{1}$$

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The first and second equations in (1) require the feasibility of x and (y, s) for (P) and (D), respectively, whereas the third equation is the so-called *complementarity condition* for (P) and (D). Note that xs indicates to the coordinate-wise (Hadamard) product of the vectors x and s.

Without loss of generality, we may assume that the problems (P) and (D) satisfy the interior-point condition (IPC) [2], i.e., there exist x^0 and (y^0, s^0) such that

$$\begin{aligned} &Ax^0 = b, \quad x^0 > 0 \\ &A^T y^0 + s^0 = c, \quad s^0 > 0. \end{aligned}$$

Let us briefly discuss about the idea behind generic primal–dual IPMs. In these methods, the third equation in (1) is replaced by a *parametric equation* $xs = \mu\mathbf{e}$, where μ is a positive parameter and $\mathbf{e}$ denotes the all-one vector, i.e., $\mathbf{e} = (1, 1, \ldots, 1)^T$. It can be easily shown that, under IPC condition and the full row rankness of A, the system obtained from (1) by replacing the complementarity condition with the parametric equation, i.e.,

$$\begin{aligned} &Ax = b, \quad x \geq 0 \\ &A^T y + s = c, \quad s \geq 0 \\ &xs = \mu\mathbf{e} \end{aligned} \tag{2}$$

has a unique solution, for each $\mu > 0$. We denote such a solution as $(x(\mu), y(\mu), s(\mu))$, and we call $x(\mu)$ as the μ-center of (P) and $(y(\mu), s(\mu))$ as the μ-center of (D). The set of all μ-centers, with $\mu > 0$, forms a homotopic path which is called the *central path* of (P) and (D) [3,4]. The central path for linear optimization was first recognized by Sonnevend [4] and Megiddo [5]. It has been proved that as $\mu \to 0$, the limit of the central path exists and converges to an analytic center of the optimal solutions set of (P) and (D).

For fixed $\mu > 0$, a direct application of the Newton method on the system (2) gives the following system for displacements Δx, Δy and Δs in order to produce the new point:

$$\begin{aligned} &A\Delta x = 0, \\ &A^T \Delta y + \Delta s = 0, \\ &x\Delta s + s\Delta x = \mu\mathbf{e} - xs. \end{aligned} \tag{3}$$

Therefore, the new iterate is computed as

$$x_+ = x + \alpha\Delta x, \qquad y_+ = y + \alpha\Delta y, \qquad s_+ = s + \alpha\Delta s$$

where $\alpha \in (0, 1]$ is obtained by using some rules so that the new iterate satisfies $(x_+, s_+) > 0$. Let us define the scaled vector v as

$$v := \sqrt{\frac{xs}{\mu}}.$$

Using the scaled vector v, the Newton system (3) can be rewritten as:

$$\begin{aligned} &\bar{A}d_x = 0 \\ &\bar{A}^T d_y + d_s = 0 \\ &d_x + d_s = v^{-1} - v \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\bar{A} := \frac{1}{\mu} AV^{-1}X = AS^{-1}V \\ &V := \operatorname{diag}(v), \qquad X := \operatorname{diag}(x), \qquad S := \operatorname{diag}(s) \\ &d_x = \frac{v\Delta x}{x}, \qquad d_s = \frac{v\Delta s}{s}. \end{aligned} \tag{5}$$

Note that $d_x = d_s = 0$ if and only if $v - v^{-1} = 0$ if and only if $x = x(\mu)$, $s = s(\mu)$.

Assuming $\psi_c(t) = \frac{t^2-1}{2} - \log t$, for $t > 0$, it can be easily seen that $\psi_c(t)$ is a strictly differentiable convex *barrier (kernel)* function on $\mathbb{R}^n_{++}$ with $\psi_c(1) = \psi_c'(1) = 0$, i.e. it attains its minimal value at $t = 1$. Note that the right hand side of the third equation in (4) is the minus gradient of the *proximity* function

$$\Psi_c(v) = \sum_{i=1}^{n} \psi_c(v_i).$$