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A numerical method for singularly perturbed turning point problems with an interior layer

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ABSTRACT

The objective of this paper is to present a numerical method for solving singularly perturbed turning point problems exhibiting an interior layer. The method is based on the asymptotic expansion technique and the reproducing kernel method (RKM). The original problem is reduced to interior layer and regular domain problems. The regular domain problems are solved by using the asymptotic expansion method. The interior layer problem is treated by the method of stretching variable and the RKM. Four numerical examples are provided to illustrate the effectiveness of the present method. The results of numerical examples show that the present method can provide very accurate approximate solutions.

1. Introduction

Singularly perturbed problems arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, optimal control, etc. These problems are characterized by the presence of a small parameter that multiplies the highest order derivative, and they are stiff and there exists a boundary or interior layer where the solutions change rapidly.

The numerical treatment of such problems present some major computational difficulties due to the presence of boundary and interior layers. Recently, a large number of special purpose methods have been developed by various authors for singularly perturbed boundary value problems [1–11]. However, discussion on the numerical solutions of singularly perturbed turning point problems is rare. Phaneendra1, Reddy and Soujanya [1] proposed a non-iterative numerical integration method on a uniform mesh for dealing with singularly perturbed turning point problems. Rai and Sharma [2–4] discussed the numerical methods for solving singularly perturbed differential-difference equations with turning points. Natesan, Jayakumar and Vigo-Aguiar [5] introduced a parameter uniform numerical method for singularly perturbed turning point problems exhibiting boundary layers. Kadalbajoo, Arora and Gupta [10] developed a collocation method using artificial viscosity for solving a stiff singularly perturbed turning point problem having twin boundary layers.

Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics, amongst other fields [12–27,11]. Recently, based on reproducing kernel theory, the authors have discussed two-point boundary value problems, nonlocal boundary value problems and partial differential equations [18–27,11]. However, it is very difficult to extend the application of reproducing kernel theory to singularly perturbed differential equations. Geng [11] developed a method for solving a class of singularly perturbed boundary value problems based on the RKM and a proper transformation. Nevertheless, this method fails to solve singularly perturbed turning point problems.

In this paper, based on the RKM presented in [12,18], an effective numerical method shall be presented for solving singularly perturbed turning point problems exhibiting an interior layer.

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Consider the following singularly perturbed problems:

$$\begin{cases} \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), & -1 < x < 1, \\ u(-1) = \alpha, \quad u(1) = \gamma, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently smooth, and such that (1.1) has a unique solution.

The solution of (1.1) exhibits a layer behavior or turning point behavior depending upon the coefficient $a(x)$. The points of the domain where $a(x) = 0$ are known as turning points. The presence of the turning point results in a boundary or interior layer in the solution of the problem and is more difficult to handle as compared to the non-turning point case. In this paper, we consider the case in which the turning point results into an interior layer in the solution of the problem.

We consider problem (1.1) with the following assumptions

$$\begin{cases} a(0) = 0, \quad a'(0) > 0, \\ b(x) \geq b_0 > 0, \quad x \in [-1, 1], \\ |a'(x)| \geq \dfrac{|a'(0)|}{2}, \quad x \in [-1, 1]. \end{cases} \tag{1.2}$$

Under these assumptions the given turning point problem possesses a unique solution exhibiting interior layers at $x = 0$. Unlike boundary layers this layer occurs in the interior of the domains and is considerably weaker.

The rest of the paper is organized as follows. In the next section, the numerical technique for (1.1) is introduced. Error analysis is introduced in Section 3. The numerical examples are given in Section 4. Section 5 ends this paper with a brief conclusion.

2. Numerical method

Following the idea of [5,6], we divide the given interval $[-1, 1]$ into three subintervals $[-1, -K\varepsilon^\rho]$, $[-K\varepsilon^\rho, K\varepsilon^\rho]$ and $[K\varepsilon^\rho, 1]$, where K, ρ are positive real numbers. The subintervals $[-1, -K\varepsilon^\rho]$ and $[K\varepsilon^\rho, 1]$ represent the regular regions, and the interval $[-K\varepsilon^\rho, K\varepsilon^\rho]$ represents the interior layer region. The asymptotic approximation technique is used to solve (1.1) in the regular regions $[-1, -K\varepsilon^\rho]$ and $[K\varepsilon^\rho, 1]$. Then the values of the asymptotic approximation are used as the boundary conditions at the so-called transition points $x = \pm K\varepsilon^\rho$. In the interior layer region $[-K\varepsilon^\rho, K\varepsilon^\rho]$, (1.1) is solved by combining the method of stretching variable and the RKM. After solving both the regular and interior layer problems their solutions are combined to obtain an approximate solution to the original problem over the entire region $[-1, 1]$.

2.1. Solutions of the regular regions problems

Consider (1.1) in right regular region $[K\varepsilon^\rho, 1]$ and left regular region $[-1, -K\varepsilon^\rho]$. Let $U_{R,M}(x)$ and $U_{L,M}(x)$ be the straightforward asymptotic expansions in the intervals $[K\varepsilon^\rho, 1]$ and $[-1, -K\varepsilon^\rho]$ respectively.

$$U_{R,M}(x) = \sum_{k=0}^{M} \varepsilon^k u_k(x), \tag{2.1}$$

where $u_k(x)$ are solutions of the following equations

$$\begin{aligned} & a(x)u_0'(x) - b(x)u_0(x) = f(x), \quad u_0(1) = \gamma, \\ & a(x)u_1'(x) - b(x)u_1(x) = -u_0''(x), \quad u_1(1) = 0, \\ & a(x)u_2'(x) - b(x)u_2(x) = -u_1''(x), \quad u_2(1) = 0, \\ & \ldots \\ & a(x)u_M'(x) - b(x)u_M(x) = -u_{M-1}''(x), \quad u_M(1) = 0. \end{aligned} \tag{2.2}$$

$$U_{L,M}(x) = \sum_{k=0}^{M} \varepsilon^k v_k(x), \tag{2.3}$$

where $v_k(x)$ are solutions of the following equations

$$\begin{aligned} & a(x)v_0'(x) - b(x)v_0(x) = f(x), \quad v_0(-1) = \alpha, \\ & a(x)v_1'(x) - b(x)v_1(x) = -v_0''(x), \quad v_1(-1) = 0, \\ & a(x)v_2'(x) - b(x)v_2(x) = -v_1''(x), \quad v_2(-1) = 0, \\ & \ldots \\ & a(x)v_M'(x) - b(x)v_M(x) = -v_{M-1}''(x), \quad v_M(-1) = 0. \end{aligned} \tag{2.4}$$