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On generalized parameterized inexact Uzawa method for a block two-by-two linear system ☆

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ABSTRACT

Recently, Chen and Jiang [F. Chen, Y.-L. Jiang, A generalization of the inexact parameterized Uzawa methods for saddle point problems, Appl. Math. Comput. 206 (2008) 765–771] presented a parameterized inexact Uzawa (**PIU**) algorithm for solving symmetric saddle point problems, where the (1, 2)- and the (2, 1)-blocks are the transpose of each other. In this paper, we extend the PIU method to the block two-by-two linear system by allowing the (1, 2)-block to be not equal to the transpose of the (2, 1)-block and the (2, 2)-block may not be zero. We prove that the iteration method is convergent under certain conditions. With different choices of the parameter matrices, we obtain several new algorithms for solving the block two-by-two linear system. Numerical experiments confirm our theoretical results and show that our method is feasible and effective.

1. Introduction

We consider the large sparse block two-by-two linear systems of the form

$$\begin{pmatrix} A & B^* \\ C & -D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is Hermitian positive definite, $D \in \mathbb{C}^{m\times m}$ is Hermitian positive semidefinite, $B, C \in \mathbb{C}^{m\times n}$ are of full rank, with $n \geq m$.

Linear systems of the form (1.1) arise in a variety of scientific and engineering applications, including computational fluid dynamics [1–4], mixed finite element approximation of elliptic partial differential equations [5,6], optimization [7–11], optimal control [12], weighted and equality constrained least squares estimation [13], stationary semiconductor device [14,15], inversion of geophysical data [16], and so on.

Recently, many techniques have been proposed for solving the linear systems (1.1), including Uzawa-type schemes [17–22], splitting methods [23–26,17,18,27–30], preconditioned Krylov subspace methods [7,31–37,1,38,11,6,39–43] and so on. For designing preconditioners and analyzing preconditioned matrices for the block two-by-two linear system (1.1), Bai et al. presented some comprehensive and systematic approaches in [31,32,23,44,33,24,34,25,36]. Also see [45] for a survey.

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For the real linear system with the special case $D = 0$ and $C = B$, i.e., the saddle point problem

$$\begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.2}$$

by using the techniques of matrix relaxation and inexact iteration, Bai, Parlett and Wang [17] established and studied the following parameterized inexact Uzawa (PIU) method.

Method 1.1 (*[17,18] The PIU Method*)**.** Let $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{m\times m}$ be symmetric positive definite matrices. Given initial vectors $x^{(0)} \in \mathbb{R}^n$ and $y^{(0)} \in \mathbb{R}^m$, and two relaxation factors ω, τ with $\omega, \tau \neq 0$. For $k = 0, 1, 2, \ldots$ until the iteration sequence $(x^{(k)}, y^{(k)})$ converges to the exact solution of the saddle point problem (1.2), compute

$$\begin{cases} x^{(k+1)} = x^{(k)} + \omega P^{-1}(b - Ax^{(k)} - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \tau Q^{-1}(B^* x^{(k+1)} - q). \end{cases}$$

Here, Q is an approximate (or preconditioning) matrix of the approximated Schur complement matrix $B^*P^{-1}B$.

Uzawa methods are of interest because they are simple, efficient and have small computer memory requirements. In [20], Chen and Jiang presented a generalization of the parameterized inexact Uzawa method for saddle point problems with $B = C$ and $D = 0$. And in [22], Zhou and Zhang studied the PIU method for the generalized saddle point problems with $B = C$ and D is symmetric positive semidefinite.

In this paper, we extend Method 1.1 to the block two-by-two linear system (1.1), obtaining the generalized parameterized inexact Uzawa method (denoted as the GPIU method). For the correspondingly obtained GPIU method, we derive sufficient conditions for guaranteeing its convergence. With different choices of the parameter matrices, we obtain several algorithms for solving the block two-by-two linear system. Numerical experiments confirm our theoretical results and show that our method is feasible and effective.

This paper is organized as follows. In Section 2, we present the GPIU method for the block two-by-two linear system. In Section 3, we prove that the iteration method is convergent under certain conditions. With different choices of the parameter matrices, in Section 4, we obtain several algorithms for solving the linear system (1.1). The numerical experiments are given in Section 5 to confirm our theoretical results and demonstrate the feasibility and effectiveness of the GPIU method. Finally, in Section 6, some brief concluding remarks are given.

2. The GPIU method

We rewrite (1.1) into the following equivalent form:

$$\mathscr{A} z \equiv \begin{pmatrix} A & B^* \\ -C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b. \tag{2.1}$$

This can be achieved by changing the sign of the last m equations in (1.1). We assume that the coefficient matrix $\mathscr{A}$ of (2.1) is nonsingular. We refer to [46] for sufficient and necessary conditions for guaranteeing the nonsingularity of the block two-by-two matrix $\mathscr{A}$ in (2.1).

For the linear system (2.1), we make the following matrix splitting:

$$\mathscr{A} := \begin{pmatrix} A & B^* \\ -C & D \end{pmatrix} = \mathcal{M} - \mathcal{N},$$

with $\mathcal{M}$ being nonsingular and of the form

$$\mathcal{M} = \begin{pmatrix} A + P_1 & 0 \\ P_2 & P_3 + D \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} P_1 & -B^* \\ P_2 + C & P_3 \end{pmatrix}, \tag{2.2}$$

where $P_1 \in \mathbb{C}^{n\times n}$ and $P_3 \in \mathbb{C}^{m\times m}$ are Hermitian positive definite matrices, $P_2 \in \mathbb{C}^{m\times n}$. Then we present the GPIU iteration method for solving the block two-by-two linear system (2.1) as follows.

Method 2.1 (*The GPIU Method*)**.** Let $P_1 \in \mathbb{C}^{n\times n}$, $P_3 \in \mathbb{C}^{m\times m}$ be Hermitian positive definite matrices, and $P_2 \in \mathbb{C}^{m\times n}$. Given initial vectors $x^{(0)} \in \mathbb{R}^n$ and $y^{(0)} \in \mathbb{R}^m$, for $k = 0, 1, 2, \ldots$ until the iteration sequence $\{x^{(k)}, y^{(k)}\}$ is convergent, compute

$$\begin{cases} x^{(k+1)} = x^{(k)} + (A + P_1)^{-1}[f - Ax^{(k)} - B^* y^{(k)}], \\ y^{(k+1)} = y^{(k)} + (P_3 + D)^{-1}[-P_2 x^{(k+1)} + (P_2 + C)x^{(k)} - Dy^{(k)} - g]. \end{cases} \tag{2.3}$$

Let $\mathcal{T} = \mathcal{M}^{-1}\mathcal{N}$ be the iteration matrix of the GPIU method (2.3) and $z = (x^T, y^T)^T,\ c = \mathcal{M}^{-1}b$. Then

$$z^{(k+1)} = \mathcal{T} z^{(k)} + c$$