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Measurement of bivariate risks by the north–south quantile points approach

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ABSTRACT

This paper attempts to determine the Value at Risk (VaR) and Conditional Value at Risk (CVaR) measures for the sum of bivariate risks under dependence. The computation of these risk measures is performed by the north–south quantile points of bivariate distributions. The Farlie–Gumbel–Morgenstern (FGM) copula model is chosen to express dependence of bivariate risks. The behaviors of VaR and CVaR are examined by varying dependence parameter values of the copula model and probability levels of the risk measures. The findings are interpreted from the view point of portfolio risk management.

1. Introduction

Risk, in general, is defined as a possible and harmful loss occurrence event and risk measures are required to reflect the vulnerability of those who are exposed to it. Risk measures are essential elements of risk management. Several quantitative risk measures that are based on the quantiles of risk distributions have been proposed and used by many authors. Artzner et al. [1], Jorion [2], Kaas et al. [3], Szegö [4] and Dhaene et al. [5] lucidly present and discuss the features and risk measurement capacities of such measures. Following them, this paper considers Value at Risk (VaR) and Conditional Value at Risk (CVaR) as measures for bivariate risks. In the finance and insurance areas there is an ample use of VaR and CVaR measures where portfolio losses, in particular, are the core subjects for decision making and risk modeling [6–8].

Claim amounts in excess of endurable limits are possible losses to cite for portfolios of insurance policies outstanding in a given time interval. Similarly, failures about returns on portfolios of assets, in a given time period, are possible losses for financial establishments. Determination of composition of such insurance and finance portfolios, pricing of insurance policies at individual or collective levels, determination of required return rates and setting of risk reserves and premiums are the decision making outputs that can be mentioned just to give some examples of the use of risk measures for.

In many circumstances it is imperative to take account of dependencies in financial and insurance portfolios: same people may exist in two different portfolios of policies of an insurer or two or more policy owners in an insurance portfolio may cause defaults interdependently. Similarly, by virtue of the presence of same sets of assets in two different financial portfolios, interdependence of return failures in portfolios comes into existence. Dependencies that arise, as exemplified, must always be expressed in risk modeling through risk measurement under dependence. Recently, copula models have been in the use of dependence modeling. Copulas are useful analytical tools that separate joint distribution of risk variables (loss amounts) into two parts: marginal distributions of individual risk variables and an expression about their joint dependence structure. Joe [9], Nelsen [10,11], Cherubini et al. [12] in a financial context, and Denuit et al. [13] in an actuarial context present a detailed discussion of copula functions.

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There is a conceptual and analytical relation between VaR, CVaR, quantiles of distributions and copulas. Chen and Welsh [14] introduce bivariate quantiles through the use of distribution functions and give multivariate extensions of quantile points. Embrechts et al. [15] use copulas to put bounds on VaR functions for sums of dependent risks and present some applications of them. Cheng et al. [16], Belzunce et al. [17], Gebizlioglu and Yagci [18], Fantazzini [19], He and Gong [20] and Halbleib and Pohlmeier [21] present some results, relevant to this paper, that connect risk measures, quantiles and copula functions.

In the following sections, CVaR function for the sum of dependent bivariate risks is determined by making use of the analytical relation between CVaR and VaR. the north–south quantile points of bivariate distributions are used up to this aim. Dependence parameters of copula functions and percentile levels of quantiles directly affect VaR and CVaR values. The VaR and CVaR functions are examined, in this regard, and interpreted by some numerical examples.

2. Bivariate north–south quantile points and risk measures

The use of bivariate north–south quantile points (BNSQP) makes it easy to perform the risk measurement by using copula functions for distributions of dependent risk variables. BNSQP approach concentrates on the quantiles of the bivariate distributions for the conduct of the inference under the bivariate analysis [14]. In this regard, we firstly refer to risk measures and copulas. Then we show the determination of the bivariate VaR and CVaR functions by using BNSQP.

Let X be a continuous random risk variable, as an individual or collective loss amount, with distribution function F. VaR at p-level for this random variable is defined as

$$\text{VaR}\,[X; p] = F_X^{-1}(p) \tag{1}$$

which is actually a p-th order quantile of F distribution. VaR is not a convex measure. In this sense we consider CVaR as a risk measure that is superior to VaR [1].

CVaR estimates the average loss amount on the tail region of a loss amount distribution beyond the VaR value. Tail risks are important for assessing excessive loss amounts. In addition, CVaR is a coherent risk measure. Any risk measure that satisfies the axioms of sub-additivity, monotonicity, positive homogeneity and translation invariance is a coherent risk [1]. It is well known that VaR is not a coherent risk measure. So it may discourage portfolio diversification. However, it is coherent for $p > 0.5$ under the assumption of elliptically distributed loss amounts when portfolio losses are linear functions of the loss values under concern [5].

CVaR is defined as follows:

$$\text{CVaR}\,[X; p] = E\,[X - \text{VaR}\,[X; p]\,|X > \text{VaR}\,[X; p]]\,.$$

CVaR is also defined by the associated mean-excess function (*mef*) as follows,

$$\text{CVaR}\,[X; p] = m_X\,(\text{VaR}\,[X; p])$$

where m_X is defined as

$$\begin{aligned} m_X(x) &= E\,[X - x|X > x]\,, \quad x > 0 \\ &= \frac{\Pi_X(x)}{\overline{F}_X(x)} \end{aligned}$$

and Π and $\overline{F}$ stand for stop-loss function and survival function of random loss variable X, respectively [13].

Now, let X and Y be continuous random loss amount variables as dependent risks, with distribution functions F_1 and F_2, respectively. Sklar [22] states that if X and Y are random variables with joint distribution function F and marginal distribution functions F_1 and F_2, a copula, C, uniquely determines $F(x, y) = C(F_1(x), F_2(y))$ on Range $F_1 \times$ Range F_2 for all x, y. Nelsen [10] presents copula techniques and several copula function families. Nelsen [11] and Rodriguez-Lallena and Ubeda-Flores [23] extend these to multivariate copula functions for more than two random variables in dependence.

The minimum and maximum copulas C^- and C^+ are named as Frechet lower and upper bounds and the inequalities $C^- \le C \le C^+$, openly expressed below

$$\max\,(F_1(x) + F_2(y) - 1, 0) \le C(F_1(x), F_2(y)) \le \min(F_1(x), F_2(y))$$

are known as the inequality of Frechet–Hoeffding for distribution functions.

The bivariate copula C is restricted to the support $I^2 = [0, 1]^2$ of the bivariate distribution function $F(x, y)$ with uniform margins on $I = [0, 1]$ such that

$$C(t, 0) = C(0, t) = 0, \qquad C(t, 1) = C(1, t) = t, \quad t \in I$$

and

$$C(\widetilde{u}_2, \widetilde{v}_2) - C(\widetilde{u}_2, \widetilde{v}_1) - C(\widetilde{u}_1, \widetilde{v}_2) + C(\widetilde{u}_1, \widetilde{v}_1) > 0$$

for all $\widetilde{u}_1, \widetilde{u}_2, \widetilde{v}_1, \widetilde{v}_2$ in I, $\widetilde{u}_1 \le \widetilde{u}_2$ and $\widetilde{v}_1 \le \widetilde{v}_2$. So, a bivariate copula with support $[0, 1]^2$ is $C\,(u_1, u_2) = F\big(F_1^{-1}\,(u_1)\,, F_2^{-1}\,(u_2)\big) = F\,(x, y)$ where u_1 and u_2 are uniformly distributed random variables.