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On periodic discrete spline interpolation: Quintic and biquintic cases

Fengmin Chen, Patricia J.Y. Wong *

School of Electrical and Electronic Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore

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ABSTRACT

In this paper, we develop a class of periodic discrete spline interpolates in one and two independent variables. Explicit error bounds are further derived for the periodic quintic and biquintic discrete spline interpolates. We also present some numerical examples to illustrate the results obtained.

1. Introduction

In continuous spline interpolation, we not only interpolate the function of interest at each knot, but also interpolate a number of derivatives of the function at certain knots. Therefore, the function is required to be sufficiently smooth. However, in real-world situations, not only may it be difficult to compute the derivatives of a function, but the derivatives may not even exist at some points. In such a situation, the usual continuous spline interpolation will not be suitable. We therefore introduce a discrete interpolation scheme that involves only differences. Since no derivatives are involved, the discrete interpolate can be constructed for a more general class of function, and therefore it has a wider range of applications.

Discrete splines are piecewise polynomials where continuity of differences rather than derivatives are satisfied at the joining points of the polynomial pieces. Discrete splines were first introduced by Mangasarian and Schumaker [1] in 1971 as solutions to constrained minimization problems in real Euclidean space, which are discrete analogs of minimization problems in Banach space whose solutions are generalized splines. Thereafter, Schumaker [2] studied constructive aspects of these discrete splines in terms of discrete *B*-splines, Astor and Duris [3] introduced discrete *L*-splines, and Lyche [4,5] discussed cubic discrete splines involving central differences. Following Lyche's research on *cubic* discrete splines involving *central differences*, many papers appeared in this area. To mention a few, in [6] the error estimate is established for the first difference of the cubic discrete spline interpolation proposed in [4,5]; in [7,8], the periodic cubic discrete spline which interpolates a given function at one interior point of each mesh interval is discussed; in [9], the cubic discrete spline satisfying certain averaging interpolatory conditions is investigated; in [10], the cubic deficient discrete spline is studied. Other related work on *non-cubic* discrete splines involving central differences includes the quadratic discrete splines discussed in [11].

Motivated by the above research, in this paper, we develop a class of *periodic quintic discrete spline interpolates* involving *central differences*, and we establish the related existence, uniqueness, and error estimates. The *two-variable* case will also be tackled. Our work naturally extends the literature and especially complements and/or extends the above-mentioned work of [7,8,4–6] on *one-variable cubic* discrete splines. We also extend the research of [12,13] from the continuous case to the discrete case, and complement the work of [14,15] on cubic and quintic discrete splines involving *forward differences*.

The outline of the paper is as follows. In Section 2, we develop periodic quintic discrete spline interpolation for a periodic function, and establish the existence and uniqueness of the discrete spline interpolate. The error analysis of the periodic

* Corresponding author. Tel.: +65 67904219; fax: +65 67933318.
E-mail addresses: chen0519@ntu.edu.sg (F. Chen), ejywong@ntu.edu.sg (P.J.Y. Wong).

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discrete spline interpolation is tackled in Section 3. Using the results of Sections 2 and 3, we discuss the two-variable case in Section 4; in particular, the periodic biquintic discrete spline interpolate of a periodic function is defined, and its existence and uniqueness as well as the error estimates are established. Finally, three numerical examples are presented in Section 5 to illustrate the interpolation procedures as well as the error estimates obtained.

2. Periodic discrete spline interpolation

For a given $h > 0$, we recall that applying the *central difference operator* D_h to a function F gives

$$D_h^{\{0\}}F(x) = F(x); \qquad D_h^{\{1\}}F(x) = \frac{F(x+h) - F(x-h)}{2h}; \qquad D_h^{\{2\}}F(x) = \frac{F(x+h) - 2F(x) + F(x-h)}{h^2};$$
$$D_h^{\{3\}}F(x) = \frac{F(x+2h) - 2F(x+h) + 2F(x-h) - F(x-2h)}{2h^3};$$
$$D_h^{\{4\}}F(x) = \frac{F(x+2h) - 4F(x+h) + 6F(x) - 4F(x-h) + F(x-2h)}{h^4}.$$

We also use the basic polynomials $x^{\{j\}}$ introduced in [5] as follows:

$$x^{\{j\}} = x^j, \quad j = 0, 1, 2; \qquad x^{\{3\}} = x(x^2 - h^2), \qquad x^{\{4\}} = x^2(x^2 - h^2), \qquad x^{\{5\}} = x(x^2 - h^2)(x^2 - 4h^2).$$

It is noted that $D_h^{\{1\}}x^{\{j\}} = jx^{\{j-1\}}$, $j = 0, 1, 2, 3, 5$, and $D_h^{\{1\}}x^{\{4\}} = 2x(2x^2 + h^2)$.

Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. We let

$$\varphi : a = t_0 < t_1 \cdots < t_n = b \quad \text{and} \quad \varphi' : c = u_0 < u_1 \cdots < u_m = d$$

denote the uniform partitions of $[a, b]$ and $[c, d]$ with step sizes $p = \frac{b-a}{n}$ and $p' = \frac{d-c}{m}$, respectively. Further, let $\phi = \varphi \times \varphi'$ be a rectangular partition of $[a, b] \times [c, d]$. Throughout, let $0 < h \le \min\{p, p'\}$ be fixed, and denote the discrete interval

$$[\alpha, \beta]_h = \{\alpha, \alpha + h, \alpha + 2h, \ldots\} \cap [\alpha, \beta].$$

We assume that p and p' are multiples of h. Then, it is clear that the t_i are in $[a, b]_h$ and the u_i are in $[c, d]_h$.

With reference to the uniform partition φ of $[a, b]$, we now develop a class of discrete spline interpolation. A similar treatment holds for the uniform partition φ' of $[c, d]$.

Definition 2.1. A function $S(t; \varphi, h)$ is called a *quintic discrete spline* if its restriction S_i on $[t_{i-1}, t_i]$ is a quintic polynomial for $i = 1, 2, \ldots, n$ and

$$D_h^{\{\mu\}}S_i(t_i) = D_h^{\{\mu\}}S_{i+1}(t_i), \quad 1 \le i \le n-1, \ \mu = 0, 1, 2, 3, 4. \tag{2.1}$$

For a positive number P_0, we say that a function g is P_0-*periodic* if

$$g(t) = g(t + P_0).$$

We now introduce the *periodic quintic* discrete spline. In the spirit of [7,8], where the *periodic cubic* discrete spline is studied, let

$$S_h(\varphi) = \Big\{S(t; \varphi, h) : S(t; \varphi, h) \text{ is a quintic discrete spline and it is } (b-a)\text{-periodic}\Big\}.$$

Definition 2.2. For a $(b - a)$-periodic function f defined on $[a - 2h, b + 2h]_h$, we say that $S_\varphi f$ is the $S_h(\varphi)$-*interpolate of* f, also known as the *periodic discrete spline interpolate of* f, if $S_\varphi f \in S_h(\varphi)$ with

$$S_\varphi f(t_i) = f(t_i) \equiv f_i, \quad 0 \le i \le n-1. \tag{2.2}$$

Remark 2.1. In Definition 2.2, it actually suffices to have the periodic function f defined on the uniform partition φ. However, for the error analysis in the next section, we require the periodic function f to be defined on $[a-2h, b+2h]_h$. To be consistent, we therefore impose throughout that the *$(b - a)$-periodic function f is defined on* $[a - 2h, b + 2h]_h$.

Let the functions g_i, $\bar{g}_i$ and $\bar{\bar{g}}_i$ satisfy the following for $0 \le i, j \le n-1$:

$$g_i(t_j) = \delta_{ij}, \quad D_h^{\{2\}}g_i(t_j) = D_h^{\{4\}}g_i(t_j) = 0,$$
$$D_h^{\{2\}}\bar{g}_i(t_j) = \delta_{ij}, \quad \bar{g}_i(t_j) = D_h^{\{4\}}\bar{g}_i(t_j) = 0,$$
$$D_h^{\{4\}}\bar{\bar{g}}_i(t_j) = \delta_{ij}, \quad \bar{\bar{g}}_i(t_j) = D_h^{\{2\}}\bar{\bar{g}}_i(t_j) = 0.$$