Journal of Computational and Applied Mathematics 255 (2014) 404–416

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A high-order compact exponential scheme for the fractional convection–diffusion equation

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ARTICLE INFO

Article history:
Received 18 January 2012
Received in revised form 12 March 2013

Keywords:
Fractional differential equation
Convection–diffusion
Compact scheme
High-order
Exponential
Error estimate

ABSTRACT

A high-order compact exponential finite difference scheme for solving the fractional convection–diffusion equation is considered in this paper. The convection and diffusion terms are approximated by a compact exponential finite difference scheme, with a high-order approximation for the Caputo time derivative. For this fully discrete implicit scheme, the local truncation error is analyzed and the Fourier method is used to discuss the stability. The error estimate is given by the discrete energy method. Numerical results are provided to verify the accuracy and efficiency of the proposed algorithm.

1. Introduction

Fractional differential equations (FDEs) have been intensely studied in the recent years, as their applications can be found in physical, biological, geological and financial systems. Detailed discussions on fractional differential equations can be found in the review article [1] and monograph [2].

Several numerical methods have been proposed for solving the space and/or time FDEs up to now, e.g., finite difference schemes for the time fractional diffusion problems were discussed in [3–6]. These schemes are convergent with order of two for the space variable(s), and it is interesting to seek high-order numerical methods for FDEs. Compact finite difference schemes have the desirable tridiagonal nature of the finite-difference equations with high-order of accuracy (see [7,8]), and one-dimensional fractional sub-diffusion equation was recently solved by the compact finite difference scheme with convergence order $\mathcal{O}(\tau + h^4)$ in [9], a higher order $\mathcal{O}(\tau^{2-\gamma} + h^4)$ one can be found in [10], and compact ADI scheme for two-dimensional problem in [11].

For steady and/or unsteady convection–diffusion equations of integer order, there have been many research papers on this topic; see e.g., [12–15]. In [14,15], the authors point out that the high-order compact exponential scheme is preferred for these kinds of equations. Therefore, we try to solve the time fractional convection–diffusion problem using the compact exponential difference scheme in this paper. The problem we consider here is a constant coefficient convection–diffusion problem, and the fractional Fokker–Planck equation is a more complicated one, with variable coefficients [16,17].

The model problem considered here is the fractional convection–diffusion equation,

$$ {}_0^C D_t^\gamma u(x,t) + p\frac{\partial u}{\partial x} - a\frac{\partial^2 u}{\partial x^2} = g(x,t), \quad L_1 < x < L_2,\ 0 < t < T. \tag{1} $$

Here the convection coefficient p is a constant, and the diffusion coefficient a is a positive constant. Note that (1) becomes the sub-diffusion equation when $p = 0$ and it was discussed in [9], so we consider the case $p \neq 0$ in this paper. The Caputo

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http://dx.doi.org/10.1016/j.cam.2013.06.001

fractional derivative ${}_0^C D_t^\gamma v (0 < \gamma < 1)$ of the function $v(x,t)$ is defined by [2], i.e.,

$$ {}_0^C D_t^\gamma v(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\frac{\partial v(x,\tau)}{\partial t}}{(t-\tau)^\gamma} d\tau. $$

The initial condition for (1) is

$$ u(x,0) = w(x), \quad L_1 < x < L_2 \tag{2} $$

with the Dirichlet boundary condition given by

$$ u(L_1,t) = \varphi_1(t), \qquad u(L_2,t) = \varphi_2(t), \quad t \ge 0. \tag{3} $$

The paper is organized as follows. In Section 2, adopting the high-order exponential (HOE) scheme for the steady problems first and then using the high order discretization for the time fractional derivative, we present an implicit compact exponential difference scheme for the fractional convection–diffusion equation. The local truncation error is discussed and the stability and convergence are analyzed using the Fourier analysis and energy method in Section 3. Finally, some numerical examples are given in Section 4 to verify the theoretical conclusions. This paper closes with a summary in Section 5.

Throughout this paper, the symbol C is a generic positive constant, it may take different values at different places. We use the "empty sum" convention $\sum_{l=p}^{q} v^l = 0$ for $q < p$.

2. High-order compact exponential difference scheme

2.1. Partition of the domain and some one-dimensional vectors

For the numerical solution of (1)–(3) we introduce a uniform grid of mesh points (x_j, t_n), with $x_j = L_1 + jh$, $j = 0, 1, \ldots, N_x + 1$, and $t_n = n\tau$, $n = 0, 1, \ldots, N$. Here N_x and N are positive integers, $h = (L_2 - L_1)/(N_x + 1)$ is the mesh-width in x, and $\tau = T/N$ is the time step. For any function $v(x,t)$, we let $v_j^n = v(x_j, t_n)$, e.g., the theoretical solution u at the mesh point (x_j, t_n) is denoted by u_j^n, and U_j^n stands for the solution of an approximating difference scheme at the same mesh point. On each time level t_n we denote the exact solution vector of order N_x by $\mathbf{u}^n = \mathbf{u}(t_n) = (u_1^n, u_2^n, \ldots, u_{N_x}^n)^T$ and the approximate solution vector $\mathbf{U}^n = \mathbf{U}(t_n) = (U_1^n, U_2^n, \ldots, U_{N_x}^n)^T$.

2.2. Derivation of the high-order exponential scheme

In order to give the compact exponential scheme for the model problem (1)–(3), we study the corresponding steady equation first, i.e., Eq. (1) without the fractional time derivative. For the ordinary differential equation

$$ p\frac{\partial u}{\partial x} - a\frac{\partial^2 u}{\partial x^2} = f(x), \tag{4} $$

a high-order compact exponential scheme for (4) has already been given in [14,15]. We state the approximate scheme here. Let U be the approximation of u in (4), the compact exponential scheme reads as

$$ (-\alpha\delta_x^2 + p\delta_x)U_j = (1 + \alpha_1\delta_x + \alpha_2\delta_x^2)f_j. \tag{5} $$

Here the difference operators δ_x and δ_x^2 are approximations for the first and second derivatives, respectively, and they are defined by

$$ \delta_x U_j = \frac{U_{j+1} - U_{j-1}}{2h}, \qquad \delta_x^2 U_j = \frac{U_{j-1} - 2U_j + U_{j+1}}{h^2}, \qquad \delta_x U_{j-\frac{1}{2}} = \frac{U_j - U_{j-1}}{h}. $$

As we discuss the case for $p \neq 0$, then the coefficients in (5) are (see [14,15])

$$ \alpha = \frac{ph}{2}\coth\left(\frac{ph}{2a}\right), \qquad \alpha_1 = \frac{a-\alpha}{p}, \qquad \alpha_2 = \frac{a(a-\alpha)}{p^2} + \frac{h^2}{6}, \tag{6} $$

then one has

$$ (-\alpha\delta_x^2 + p\delta_x)u_j = (1 + \alpha_1\delta_x + \alpha_2\delta_x^2)f_j + O(h^4), $$

and the local truncation error for this compact exponential scheme is $O(h^4)$.

Now we turn to the numerical solution of (1). Similarly to what have been done in [13,15], this HOE scheme can be extended directly to the unsteady problem (1) by simply replacing $u(x)$ by $u(x,t)$, and $f(x)$ by $-{}_0^C D_t^\gamma u(x,t) + g(x,t)$. Therefore