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A posteriori error analysis for discontinuous finite volume methods of elliptic interface problems

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ABSTRACT

This paper presents two types of a posterior error estimations for discontinuous finite volume methods to elliptic interface problems. One is the residual-based estimator and other one is the recovery-based estimator. For both a posterior error estimators, we establish reliability and efficiency bounds. Numerical experiments are conducted to validate our theoretical conclusions.

1. Introduction

The finite volume method (FVM) is a discretization technique for solving partial differential equations. Due to the property of local conservation, FVM is widely used in many fields such as computational fluid dynamics.

The notion of a posteriori error estimate and adaptive mesh refinement was first introduced into finite element analysis in 1978 by Babuška and Rheinboldt [1,2]. A posteriori error estimate plays an important role in adaptive mesh refinement procedure. It uses computable quantities such as a numerical solution to estimate the error between the exact solution and the numerical solution. The computable a posteriori error estimator can be used as an indicator to guide mesh refinement to reduce the global error. There are two kinds of a posteriori error estimators: one is the residual-based error estimator (see [3–11]) and other is the recovery-based error estimator (see [12–14]). The recovery-based methods have been widely used in engineering, and studied by many researchers, for example ([15–22]), because of their many appealing features. The recent work of a posteriori error estimate for finite volume methods can be found in [23–28].

In this paper, we will study residual-based and recover-based a posteriori error estimators for the discontinuous finite volume method for elliptic interface problems. The numerical schemes of discontinuous finite volume methods have been introduced in [29,28]. We obtain reliability and efficiency bounds for both the residual-based and recovery-based estimators. Our numerical experiments show that these estimators can be used as an effective indicator to reduce the error and to catch the singularity of problems. A posteriori error analysis is investigated for discontinuous finite element methods for the elliptic interface problem in [30]. This paper is motivated by the work in [30].

The remainder of this article is organized as follows. In Section 2, the elliptic interface problem and its discontinuous finite volume formulation are introduced. In Section 3 we introduce two interpolation operators and their properties respectively. Then in Section 4, the residual-based a posteriori error estimation is analyzed. The recovery procedure and the recovery-based a posteriori error estimator are established in Section 5. Finally, in Section 6, we provide numerical results to support our conclusions.

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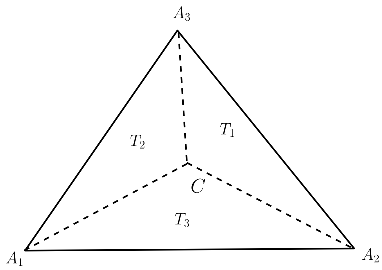

Fig. 1. Primal and dual partition.

1.1. Preliminaries and the problems

In this paper, let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider the following interface problem

$$-\nabla\cdot(k(\mathbf{x})\nabla u) = f \quad \text{in } \Omega \tag{1}$$

with Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where f is a given scalar-valued function, and $k(\mathbf{x})$ is positive and piecewise constant on polygonal subdomains of Ω with possible large jumps across subdomain boundaries (interfaces):

$$k(\mathbf{x}) = k_i > 0 \quad \text{in } \Omega_i$$

for $i = 1, \ldots, n$. Here, $\{\Omega_i\}_{i=1}^n$ is a partition of the domain Ω with Ω_i being an open polygonal domain. Let $k_{\max} = \max_{x\in\Omega} k(\mathbf{x})$ and $k_{\min} = \min_{x\in\Omega} k(\mathbf{x})$.

This paper will use standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$ and seminorms $|\cdot|_{s,D}$, for $s \geq 0$, where D is a bounded polygonal domain. The space $H^0(D)$ coincides with $L^2(D)$, with inner products $(\cdot,\cdot)_D$. When $D = \Omega$, we will use $(\cdot,\cdot)$ to denote the inner product.

Next, we define two operators. For a vector-valued function $\boldsymbol{\tau} = (\tau_1, \tau_2)^t$, define the divergence operator as follows

$$\nabla\cdot\boldsymbol{\tau} := \frac{\partial\tau_1}{\partial x_1} + \frac{\partial\tau_2}{\partial x_2}.$$

For a scalar-valued function v, define the operator $\nabla^\perp$ by

$$\nabla^\perp v = \left(-\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1}\right)^t.$$

Moreover, another Hilbert space is defined as

$$H(div;\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^2 : \nabla\cdot\tau \in L^2(\Omega)\}$$

equipped with the norm

$$\|\boldsymbol{\tau}\|_{H(div;\Omega)} = \left(\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\nabla\cdot\boldsymbol{\tau}\|_{0,\Omega}^2\right)^{\frac{1}{2}}.$$

Let $\mathcal{T}_h$ be a finite element triangulation of the domain Ω. Assume that the triangulation $\mathcal{T}_h$ is regular; i.e. $diam(K) \leq \kappa\rho_K,\ \forall K \in \mathcal{T}_h$. Here $diam(K)$ is the diameter of the element K and ρ_K denotes the diameter of the largest inscribed circle of K. Moreover, we will assume the interfaces do not cut through any element $K \in \mathcal{T}_h$. For a given triangulation $\mathcal{T}_h$, its dual partition $\mathcal{T}_h^*$ is the union of smaller triangles T_1, T_2, and T_3. These smaller triangles are formed by connecting the barycenter and the three corners of the triangles (shown as Fig. 1 $T_1 = \Delta A_2CA_3$, $T_2 = \Delta A_3CA_1$, $T_3 = \Delta A_1CA_2$).

The trial function space associated with $\mathcal{T}_h$ for the discontinuous finite volume method is defined as

$$V_h = \{v \in L^2(\Omega) : v|_K \in P_1(K), \forall K \in \mathcal{T}_h\}.$$

The test function space is defined by

$$Q_h = \{p \in L^2(\Omega) : p|_T \in P_0(T), \forall T \in \mathcal{T}_h^*\}.$$

Let $\mathcal{E}_h$ be the union of edges of triangles $K \in \mathcal{T}_h$, and $\mathcal{E}_h^0 := \mathcal{E}_h \setminus \partial\Omega$ be the interior edges. Similarly, for $e \in \mathcal{E}_h$, we can define $(\cdot,\cdot)_{s,e}$ and $\|\cdot\|_{s,e}$ with $s \geq 0$.