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Conditional gradient Tikhonov method for a convex optimization problem in image restoration

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ABSTRACT

In this paper, we consider the problem of image restoration with Tikhonov regularization as a convex constrained minimization problem. Using a Kronecker decomposition of the blurring matrix and the Tikhonov regularization matrix, we reduce the size of the image restoration problem. Therefore, we apply the conditional gradient method combined with the Tikhonov regularization technique and derive a new method. We demonstrate the convergence of this method and perform some numerical examples to illustrate the effectiveness of the proposed method as compared to other existing methods.

1. Introduction

Image restoration is one of the classical problems in image processing. The problem consists of the reconstruction of an original image that has been digitized and has been degraded by a blur and an additive noise. The blurring process is described mathematically by a *point spread function* (PSF), which is a function that specifies how pixels in the image are distorted. The discrete model of the image restoration problem can be presented as the following system of linear equations

$$g = Ax + \mathbf{n}, \tag{1}$$

where x, $\mathbf{n}$ and g are np-vectors representing the true image X of size $n \times p$, the distorted image G and the additive noise $\mathbf{N}$, respectively; the vectors x, g and $\mathbf{n}$ are obtained by stacking the rows of X, G and $\mathbf{N}$, respectively. The matrix A is an $np \times np$ matrix that represents the blurring phenomena. It is constructed from the PSF, and is called the blurring matrix. It is well known that the blurring matrix A is of ill-determined rank, i.e., A is very large and has many singular values of different orders of magnitude close to the origin. The right-hand side vector g in (2) represents the available output and is assumed to be contaminated by an error (noise) $\mathbf{n}$, i.e., $g = \widehat{g} + \mathbf{n}$, where $\widehat{g} = A\widehat{x}$ is the noise-free degraded image and $\widehat{x}$ is the vector representing the true image $\widehat{X}$. The goal of the restoration problem is to obtain an acceptable approximation to the original image. It is well-known that image restoration problems are converted into large-scale ill-posed problems. Some treatments and overviews on image restoration can be found in [1–3]. In this paper, we restrict our attention to the problem (2) involving image restoration. Here, we consider the convex minimization problem

$$\min_{x \in \widetilde{\Omega}} \|Ax - g\|_2. \tag{2}$$

The set $\widetilde{\Omega} \subset \mathbb{R}^{np}$ could be a simple convex set (e.g., a sphere or a box) or the intersection of some simple convex sets. Due to the ill conditioning of the problem (2), we replace the original problem by a better conditioned one in order to diminish

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the effects of the noise in the data. One of the most popular regularization methods is due to Tikhonov. The method replaces the problem (2) with the new one

$$\min_{x\in\widetilde{\Omega}}(\|Ax-g\|_2^2+\lambda^2\|Rx\|_2^2), \tag{3}$$

where R is a regularization operator chosen to obtain a solution with desirable properties such as small norm or good smoothness.

Image restoration problems with constraints or large-scale linear discrete ill-posed problems with constraints have been widely studied in the literature. Several different approaches to their solution have been proposed; see, e.g., [4–9]. To the best of our knowledge, combining the conditional gradient method with the regularization Tikhonov techniques, for solving the image restoration problem, has not previously received any attention in the literature. In this paper, we consider the case when both the blurring matrix and the Tikhonov regularization matrix are factorized as Kronecker products of two matrices.

In this paper, we combine the conditional gradient algorithm with the Tikhonov regularization technique to obtain a new method that will be applied to image restoration problems. The outline of this paper is as follows. In Section 2, we will consider the problem (3) for image restoration as a matrix convex optimization problem. The blurring matrix A will be given as a Kronecker product of two small matrices. Using some appropriate properties of the Kronecker product, the convex optimization problem to treat here will be of a reasonable dimension. To estimate an optimal value of the regularization parameter λ, we will use the generalized cross-validation method based on the generalized singular value decomposition (GSVD). In Section 3, we propose the conditional gradient algorithm to solve the obtained convex optimization problem. The conditional gradient method is combined with the Tikhonov regularization and leads to a new method called the conditional gradient-Tikhonov method. We also give some convergence results of the proposed process. Finally, in Section 4, we give some numerical examples to illustrate the effectiveness of our proposed method in image restoration and compare our proposed method to the Reduced Newton method proposed in [10].

2. Matrix convex Tikhonov minimization problem

In practice, the typical values of n and p are 256, 512, and 1024, so the dimensions of the blurring matrix A are extremely large and a direct solution of the problems (2) and (3) involves the manipulation of large systems of simultaneous equations whose solution is beyond the capabilities of most present-day computers.

The Kronecker product plays an important role in image processing. Particularly, sparse factorization of the blurring matrix by the Kronecker product of Toeplitz matrices provides practical algorithms. Let $A=(a_{ij})$ be an $n\times p$ matrix and $B=(b_{ij})$ be an $s\times q$ matrix. The Kronecker product of the matrices A and B is defined as the $(ns)\times(pq)$ matrix $A\otimes B=(a_{ij}B)$. The vec operator transforms the matrix A to a vector $\mathbf{a}$ of size $np\times 1$ by stacking the rows of A, namely,

$$\mathbf{a}=\mathrm{vec}(A):=(a_{11},\ldots,a_{1p},a_{21},\ldots,a_{2p},\ldots,a_{n1},\ldots,a_{np})^T.$$

We will use the following property of the Kronecker product given in [11],

$$\mathrm{vec}(AXB)=(B^T\otimes A)\mathrm{vec}(X). \tag{4}$$

For A and B two matrices in $\mathbb{R}^{n\times p}$, we define the following inner product $\langle A|B\rangle_F=\mathrm{tr}(A^T B)$ where $\mathrm{tr}(Z)$ denotes the trace of the square matrix Z and A^T is the transpose of the matrix A. It follows that the well known Frobenius norm denoted by $\|.\|_F$ is given by $\|A\|_F=\sqrt{\langle A|A\rangle_F}$. We have the following property

$$\langle A|B\rangle_F=\langle\mathrm{vec}(A)|\mathrm{vec}(B)\rangle_2,\qquad \|A\|_F=\|\mathrm{vec}(A)\|_2,$$

where $\langle .\mid .\rangle_2$ denotes the l_2-inner product.

A separable PSF is often assumed in many applications [1]. In the context of image restoration when the point spread function (PSF) is separable the blurring matrix A given in (1) can be approximated as a Kronecker product $A=A_2\otimes A_1$ of two blurring matrices of appropriate sizes. In the case of nonseparable PSF, one can solve the Kronecker product approximation problem (KPA). The KPA consists of the following mean-squares problem

$$(\widehat{A}_1,\widehat{A}_2)=\arg\min_{A_1,A_2}\|A-A_2\otimes A_1\|_F.$$

Van Loan and Pitsianis [12] show how to solve the KPA problem using the singular value decomposition (SVD). Recently, Kamm and Nagy [13,14] give an efficient algorithm for computing a solution of KPA problem in image restoration. In this paper, the hypothesis of a separable PSF together with an additive white Gaussian noise are assumed. So, we assume that $A=A_2\otimes A_1$ and $R=R_2\otimes R_1$ where A_1,R_1 are square matrices of dimension $n\times n$ and A_2,R_2 are square matrices of dimension $p\times p$. The problem (3) can be rewritten as

$$\min_{x\in\widetilde{\Omega}}(\|(A_2\otimes A_1)x-g\|_2^2+\lambda^2\|(R_2\otimes R_1)x\|_2^2). \tag{5}$$