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Generalized Hermite spectral method matching asymptotic behaviors

Chao Zhang [a], Ben-yu Guo [b,c,d,*]

[a] Jiangsu Normal University, Xuzhou, 221116, China
[b] Shanghai Normal University, Shanghai, 200234, China
[c] Scientific Computing Key Laboratory of Shanghai Universities, Shanghai, 200234, China
[d] Division of Computational Science of E-institute of Shanghai Universities, Shanghai, 200234, China

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ABSTRACT

In this paper, we propose the generalized Hermite spectral method by using a family of new generalized Hermite functions, which are mutually orthogonal with the weight function $(1 + x^2)^{-\gamma}$, γ being an arbitrary real number. We establish the basic results on the corresponding orthogonal approximation and interpolation, which simulate the asymptotic behaviors of approximated functions at infinity reasonably. As examples of applications, the spectral schemes are provided for two model problems. Numerical results demonstrate their spectral accuracy in space.

1. Introduction

During the past two decades, more and more attentions were paid to numerical solutions of differential equations defined on unbounded domains. For solving problems defined on the whole line and the related unbounded domains, we may use the Hermite orthogonal approximation and the Hermite–Gauss interpolation. Guo [1], and Guo and Xu [2] developed the spectral and pseudospectral methods for nonlinear partial differential equations, by using the standard Hermite polynomials which are mutually orthogonal with the weight function e^{-x^2}. Weideman [3] presented the related implementations. These methods are also available, even if the approximated solutions grow like $e^{\alpha x^2}(\alpha < \frac{1}{2})$ as $|x|$ increases. However, the small global numerical errors with the weight function e^{-x^2} do not imply the small point-wise numerical errors for large $|x|$ automatically. Meanwhile, Funaro and Kavian [4] considered the spectral method for linear parabolic equations by using the orthogonal system with the weight function $e^{\gamma x^2}(\gamma > 0)$. Fok, Guo and Tang [5] applied a similar approach coupled with finite difference approximation, to the simplified Fokker–Planck equation. Such methods are only suitable for problems with solutions behaving like $e^{-\alpha x^2}(\alpha > \frac{1}{2}\gamma)$ at infinity. On the other hand, Guo, Shen and Xu [6] provided the spectral and pseudospectral methods for the Dirac equation with the solution behaving like $(1 + x^2)^{-\frac{1}{2}\alpha}(\alpha > 1)$ at infinity, by using the Hermite functions which are mutually orthogonal with the weight function 1. We also refer the readers to the work of

* Corresponding author at: Shanghai Normal University, Shanghai, 200234, China. Tel.: +86 21 64322851; fax: +86 21 64323364.
E-mail addresses: zcxz1977@163.com (C. Zhang), byguo@shnu.edu.cn (B.Y. Guo).

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Boyd [7,8], Ma, Sun and Tang [9], Ma and Zhao [10], and Xiang and Wang [11]. In many practical problems, the solutions might behave like $(1+x^2)^{\frac{1}{2}\alpha}$ for large $|x|$, α being certain real number. In these cases, it seems reasonable to adopt the orthogonal approximation with the weight function like $(1+x^2)^{-\gamma}$, $\gamma > \alpha + \frac{1}{2}$.

In this paper, we propose a family of new generalized Hermite functions, which are mutually orthogonal with the weight function $(1+x^2)^{-\gamma}$, γ being any real number. We establish the basic results on the corresponding Hermite orthogonal approximation and Hermite–Gauss interpolation. By adjusting the parameter γ suitably, such approximations might simulate the asymptotic behaviors of approximated functions at infinity reasonably, and so play an important role in the related Hermite spectral and pseudospectral methods for differential equations with various asymptotic behaviors at infinity. As examples of applications, we provide the spectral schemes for a linear model problem and the sine–Gordon equation, and prove their spectral accuracy in space. The numerical results demonstrate the effectiveness of the suggested algorithms.

This paper is organized as follows. The next section is for preliminaries. In Section 3, we introduce the new generalized Hermite orthogonal approximation and Hermite–Gauss interpolation. In Section 4, we propose the spectral schemes for two model problems, and present some numerical results. The final section is for concluding remarks.

2. Preliminaries

In this section, we recall some results on the existing Hermite orthogonal approximation and Hermite–Gauss interpolation.

Let $\Lambda = \{x \mid -\infty < x < \infty\}$ and $\chi(x)$ be a certain weight function. For any integer $r \geq 0$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, with the inner product $(\cdot,\cdot)_{r,\chi,\Lambda}$, the semi-norm $|\cdot|_{r,\chi,\Lambda}$ and the norm $\|\cdot\|_{r,\chi,\Lambda}$. In particular, the inner product and the norm of $L^2_\chi(\Lambda)$ are denoted by $(\cdot,\cdot)_{\chi,\Lambda}$ and $\|\cdot\|_{\chi,\Lambda}$, respectively. We omit the subscript χ in notations whenever $\chi(x) \equiv 1$. For simplicity of statements, we denote $\frac{d^k v}{dx^k}$ by $\partial_x^k v$, etc.

Let $H_l(x)$ be the standard Hermite polynomial of degree l. For any $\beta > 0$, the scaled Hermite functions are given by

$$H_l^\beta(x) = \frac{1}{\sqrt{2^l l!}} e^{-\frac{1}{2}\beta^2 x^2} H_l(\beta x), \quad l \geq 0.$$

They are the eigenfunctions of the following singular Sturm–Liouville problem,

$$e^{\frac{1}{2}\beta^2x^2}\partial_x\left(e^{-\beta^2x^2}\partial_x\left(e^{\frac{1}{2}\beta^2x^2}v(x)\right)\right) + \lambda_l^\beta v(x) = 0, \quad \lambda_l^\beta = 2\beta^2 l, \ l \geq 0. \tag{2.1}$$

Let $\delta_{l,m}$ be the Kronecker symbol. The set of all $H_l^\beta(x)$ is a complete $L^2(\Lambda)$-orthogonal system, i.e.,

$$\int_\Lambda H_l^\beta(x) H_m^\beta(x) dx = \frac{\sqrt{\pi}}{\beta}\delta_{l,m}. \tag{2.2}$$

For any $v \in L^2(\Lambda)$, we have

$$v(x) = \sum_{l=0}^\infty v_l^\beta H_l^\beta(x), \tag{2.3}$$

with

$$v_l^\beta = \frac{\beta}{\sqrt{\pi}}\int_\Lambda v(x) H_l^\beta(x) dx.$$

Let

$$\mathcal{Q}_N^\beta(\Lambda) = \operatorname{span}\{H_l^\beta(x), \ 0 \leq l \leq N\}.$$

The $L^2(\Lambda)$-orthogonal projection $P_{N,\beta,\Lambda} : L^2(\Lambda) \to \mathcal{Q}_N^\beta(\Lambda)$ is defined by

$$(P_{N,\beta,\Lambda} v - v, \phi) = 0, \quad \forall \phi \in \mathcal{Q}_N^\beta(\Lambda). \tag{2.4}$$

For any integer $r \geq 0$, we define the space

$$H^r_{A,\beta}(\Lambda) = \{u \mid \|u\|_{H^r_{A,\beta}(\Lambda)} < \infty\},$$

equipped with the norm

$$\|u\|_{H^r_{A,\beta}(\Lambda)} = \left(\sum_{k=0}^r \left\|(\beta^4 x^2 + \beta^2)^{\frac{r-k}{2}}\partial_x^k u\right\|_\Lambda^2\right)^{\frac{1}{2}}.$$

Throughout this paper, we denote by c a generic positive constant independent of any function, N and β. According to Theorem 2.1 of [11], we have the following result.