Journal of Computational and Applied Mathematics 255 (2014) 661–670

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Convergence analysis of moving finite element methods for space fractional differential equations

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ARTICLE INFO

Article history:
Received 10 February 2013
Received in revised form 11 June 2013

MSC:
65M60
65M12
65M06
35S10
65R20

Keywords:
Finite element methods
Fractional differential equations
Convergence analysis

ABSTRACT

Most existent papers have focused on the fixed mesh methods for solving space fractional differential equations. However since some classes of space fractional differential equations may have singular or even finite-time blowup solutions, it is highly needed to develop adaptive mesh methods to solve these problems. In this paper the moving finite element methods are studied for a class of time-dependent space fractional differential equations. The convergence theories of the methods are derived with L^2-norm and numerical examples are provided to support the theoretical results. To simplify the analysis, a fractional Ritz projection operator is introduced and the error estimation of the projection is derived under the moving mesh setting.

1. Introduction

Consider a time-dependent space fractional differential equation of the following form

$$u_t - D(p\,{}_aD_x^{-\beta} + q\,{}_xD_b^{-\beta})Du = f, \quad x \in \Omega := (a, b),\ t \in I := (0, T], \tag{1}$$

$$u(a, t) = 0, \qquad u(b, t) = 0, \quad t \in I, \tag{2}$$

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \tag{3}$$

where $0 < \beta < 1$, f and φ are given functions, D denotes a single partial derivative, ${}_aD_x^{-\beta}$ and ${}_xD_b^{-\beta}$ represent left and right fractional integral operators, p and q are two constants satisfying that $0 \le p, q \le 1$ and $p + q = 1$.

The above equation has many applications in population growth models (see e.g., [1,2]). For some nonlinear reaction terms $f = f(x, t, u)$, the above equation has a finite-time blowup solution which means that the solution tends to infinity as time approaches to a finite time (see e.g., [3]). Since numerical methods on fixed meshes cannot efficiently solve the above nonlinear problems, adaptive mesh methods need to be developed. The moving mesh method as one of adaptive mesh methods has great advantages in solving blowup problems (see e.g., [4–9]). These motivate us to develop moving mesh methods to solve the fractional differential equations.

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0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.06.021

There are a vast number of references for developing and analyzing numerical methods on fixed mesh for solving fractional differential equations (see e.g., [10–27] which focus on finite difference methods; [28–41] on Galerkin methods or finite element methods).

There are a few of references for developing moving mesh methods for fractional differential equations. Ma and Jiang [8] developed moving mesh collocation methods to solve nonlinear time fractional partial differential equations with blowup solutions. However the analysis was not provided by the paper [8]. Although moving mesh methods have been well developed (see e.g., the books [42,9]), the convergence analyses have not been fully understood, hitherto having only focused on integer differential equations (see e.g., [43–50]). Jiang and Ma [51] analyzed moving mesh finite element methods for time fractional partial differential equations and simulated the blowup solutions. In general fractional differential equations are classified into three classes—time fractional differential equations, space fractional differential equations and space–time fractional differential equations. The analyses of the numerical methods are essentially and technically different for different kinds of fractional differential equations.

In this paper we study the convergence rate of moving finite element methods for space fractional differential equations with focus on linear reaction term $f = f(x,t)$. We prove the convergence rate in the L^2 norm and provide numerical examples to support the theoretical results. To simplify the proof we introduce a fractional Ritz projection operator, which is analogous to the standard Ritz projection operator in [52], and prove the error estimations of the fractional Ritz projection under a moving mesh setting.

Throughout the paper we use notation $g_1 \lesssim g_2$ and $g_1 \gtrsim g_2$ to denote $g_1 \le Cg_2$ and $g_1 \ge Cg_2$, respectively, where C is a generic positive constant independent of any functions and numerical discretization parameters.

2. Variational forms

Define the left Riemann–Liouville fractional integral as

$$ {}_aD_x^{-\sigma}u(x) = \frac{1}{\Gamma(\sigma)}\int_a^x (x-\xi)^{\sigma-1}u(\xi)\,d\xi, \quad x > a,\ \sigma > 0, \tag{4}$$

where $a \in \mathbb{R}$ or $a = -\infty$, and right Riemann–Liouville fractional integral as

$$ {}_xD_b^{-\sigma}u(x) = \frac{1}{\Gamma(\sigma)}\int_x^b (\xi-x)^{\sigma-1}u(\xi)\,d\xi, \quad x < b,\ \sigma > 0, \tag{5}$$

where $b \in \mathbb{R}$ or $b = +\infty$. The Caputo left and right fractional derivatives are defined by, respectively,

$$ {}_aD_x^{\mu}u(x) = {}_aD_x^{-\sigma}D^n u(x), \quad \sigma = n-\mu,\ n-1 \le \mu < n, \tag{6}$$

$$ {}_xD_b^{\mu}u(x) = {}_xD_b^{-\sigma}D^n u(x), \quad \sigma = n-\mu,\ n-1 \le \mu < n. \tag{7}$$

Define three functional spaces $J_{L,0}^{\mu}(\Omega),\ J_{R,0}^{\mu}(\Omega),\ H_0^{\mu}(\Omega),\ \mu > 0$ as the closures of $C_0^{\infty}(\Omega)$ under the respective norms

$$ \|u\|_{J_L^{\mu}(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + \|{}_aD_x^{\mu}u\|_{L^2(\Omega)}^2\right)^{1/2}, \tag{8}$$

$$ \|u\|_{J_R^{\mu}(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + \|{}_xD_b^{\mu}u\|_{L^2(\Omega)}^2\right)^{1/2}, \tag{9}$$

$$ \|u\|_{H^{\mu}(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + \||\omega|^{\mu}\mathcal{F}(\widetilde{u})\|_{L^2(\mathbb{R})}^2\right)^{1/2}, \tag{10}$$

where $\mathcal{F}(\widetilde{u})$ denotes the Fourier transform of $\widetilde{u}$, $\widetilde{u}$ is the extension of u by zero outside of Ω.

Let $\alpha := 1 - \beta/2$. Then the variational form of problem (1) with boundary conditions (2) and initial condition (3) is defined as: Find $u \in H_0^{\alpha}(\Omega)$ such that

$$ (u_t, v) + B(u,v) = F(v), \quad \forall v \in H_0^{\alpha}(\Omega), \tag{11}$$

$$ (u(x,0), v) = (\varphi(x), v), \quad \forall v \in H_0^{\alpha}(\Omega), \tag{12}$$

where

$$ \begin{aligned} B(u,v) &:= p\left\langle {}_aD_x^{-\beta}Du, Dv\right\rangle + q\left\langle {}_xD_b^{-\beta}Du, Dv\right\rangle \\ &= p\left\langle {}_aD_x^{-\beta/2}Du,\ {}_xD_b^{-\beta/2}Dv\right\rangle + q\left\langle {}_xD_b^{-\beta/2}Du,\ {}_aD_x^{-\beta/2}Dv\right\rangle \\ &= p\left\langle {}_aD_x^{\alpha}u,\ {}_xD_b^{\alpha}v\right\rangle + q\left\langle {}_xD_b^{\alpha}u,\ {}_aD_x^{\alpha}v\right\rangle, \end{aligned} \tag{13}$$

where the derivation of the last two identities can be seen from [29], and

$$ F(v) := \langle f, v\rangle, \tag{14}$$

where (u, v) denotes L_2 inner product, $\langle\cdot,\cdot\rangle$ denotes the duality pairing of $H^{-\mu}(\Omega)$ and $H_0^{\mu}(\Omega)$, $\mu \ge 0$.

The properties of the bilinear form $B(\cdot,\cdot)$ are given by the following Lemma 2.1 whose proof can be found in [29].