Journal of Computational and Applied Mathematics 255 (2014) 671–683

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Optimal reinsurance–investment problem for maximizing the product of the insurer's and the reinsurer's utilities under a CEV model

Danping Li [a], Ximin Rong [a,b], Hui Zhao [a,*]

[a] School of Science, Tianjin University, Tianjin 300072, PR China

[b] Center for Applied Mathematics, Tianjin University, Tianjin 300072, PR China

ARTICLE INFO

Article history:
Received 11 April 2013
Received in revised form 15 June 2013

Keywords:
Proportional reinsurance
Optimal investment
Product of exponential utilities
Constant elasticity of variance (CEV) model
Hamilton–Jacobi–Bellman (HJB) equation

ABSTRACT

This paper studies the optimal reinsurance and investment problem considering both an insurer's utility and a reinsurer's utility. The claim process is assumed to follow a Brownian motion with drift, and the insurer can purchase proportional reinsurance from the reinsurer. The insurer is allowed to invest in a risk-free asset and a risky asset whose price satisfies the constant elasticity of variance (CEV) model. In addition, the reinsurer is allowed to invest in a risk-free asset to reduce the risk. Taking both the insurer and the reinsurer into account, this paper aims to maximize the expected product of the insurer's and the reinsurer's exponential utilities of terminal wealth. By solving the corresponding Hamilton–Jacobi–Bellman (HJB) equation, we derive the optimal reinsurance and investment strategies explicitly. Furthermore, we discuss the properties of the optimal strategies analytically. Finally, numerical simulations are presented to illustrate the effects of model parameters on the optimal strategies.

1. Introduction

Recently, reinsurance and investment have become more and more important in the insurance business, and they have attracted a great deal of interest. For example, Browne [1], Hipp and Plum [2], Schmidli [3], Promislow and Young [4], and Luo et al. [5] studied the optimal investment or reinsurance and investment problem of minimizing the ruin probability. They obtained optimal strategies for the diffusion risk model or compound Poisson risk model explicitly.

Besides ruin probability minimization, expected exponential utility maximization is another important objective function, and it has inspired a lot of research. Browne [1] used the diffusion risk model and found the optimal investment strategy of exponential utility maximization. Yang and Zhang [6] considered the optimal exponential utility maximization problem for an insurer with the jump–diffusion risk process. Wang [7] derived the optimal investment strategy to maximize the expected exponential utility of terminal wealth with multiple risky assets. Wang et al. [8] solved the optimization problem of exponential utility maximization via the martingale approach. Bai and Guo [9] studied the optimal reinsurance and investment problem of maximizing the expected exponential utility with multiple risky assets. Xu et al. [10] investigated a similar problem to Bai and Guo [9] for a perturbed classical risk model. Cao and Wan [11] obtained the reinsurance–investment strategy for exponential utility maximization.

In most of the literature mentioned above, the risky asset's price is supposed to follow a geometric Brownian motion (GBM), which implies that the volatility of the risky asset's price is constant and deterministic. But empirical evidence of fluctuating historical volatility shows that this is contrary to practice. It is clear that the constant elasticity of variance (CEV)

* Corresponding author. Tel.: +86 22 2740 3424; fax: +86 22 2740 3425.
E-mail addresses: lidanping@tju.edu.cn (D. Li), rongximin@tju.edu.cn (X. Rong), zhaohuimath@tju.edu.cn (H. Zhao).

0377-0427/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cam.2013.06.033

model with stochastic volatility is more practical. The CEV model proposed by Cox and Ross [12] is a natural extension of the GBM model. This model has the ability of capturing the implied volatility skew and explaining the volatility smile. It was usually applied to pricing options, analyzing the sensitivities and implied volatilities of options, as was done by Davydov and Linetsky [13]. Recently, Xiao et al. [14] and Gao [15,16] began to apply the CEV model to the pension investment problem. Gu et al. [17] used the CEV model to study a reinsurance–investment problem for an insurer with the diffusion risk model. Liang et al. [18] and Lin and Li [19] investigated the proportional reinsurance and investment problem for the jump–diffusion risk model under the CEV model. Gu et al. [20] studied the excess-of-loss reinsurance and investment problem under the CEV model and obtained the explicit optimal strategy of exponential utility maximization.

However, the above research only considers the asset management of the insurer; it ignores the profit of the reinsurer. Thus we focus on an optimization problem for both the insurer and the reinsurer in this paper. In our model, the basic claim process is assumed to follow a Brownian motion with drift, and the insurer can purchase proportional reinsurance from the reinsurer. The insurer is allowed to invest in a risk-free asset and a risky asset whose price is described by the CEV model. Moreover, the reinsurer is allowed to invest in a risk-free asset, and we aim to maximize the expected product of the insurer's and the reinsurer's exponential utilities of terminal wealth. By using stochastic control theory, we establish the corresponding Hamilton–Jacobi–Bellman (HJB) equation, and we solve it by employing a power transform and a variable change technique suggested by Cox [21]. Optimal reinsurance and investment strategies are obtained explicitly, and the properties of the strategies are analyzed. On the one hand, due to the consideration of the reinsurer's utility of terminal wealth, the optimal reinsurance strategy we obtained not only depends on the insurer's risk aversion coefficient, but also depends on the reinsurer's risk aversion coefficient. On the other hand, the optimal investment strategy we derived under the CEV model contains two parts. The first part is akin to the optimal strategy under the GBM model, except in this part we have a stochastic volatility. The second part can be explained as a correction factor resulted from the changes of the volatility under a CEV model. Furthermore, we provide a numerical analysis and find that the effect of the insurer's risk aversion coefficient on the optimal reinsurance strategy depends on the reinsurer's risk aversion coefficient, and vice versa.

This paper proceeds as follows. In Section 2, we introduce the formulation of our model. Section 3 provides the optimal proportional reinsurance and investment strategies of maximizing the expected product of the insurer's and the reinsurer's exponential utilities. In Section 4, numerical simulations are presented to illustrate our results. Section 5 concludes this paper.

2. Model formulation

We model the claim process $C(t)$ according to a Brownian motion with drift as

$$\mathrm{d}C(t) = a\mathrm{d}t - b\mathrm{d}W_0(t), \tag{2.1}$$

where a and b are positive constants and $W_0(t)$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. $(\mathcal{F}_t)$ is an augmented filtration generated by the Brownian motions with $\mathcal{F} = \mathcal{F}_T$, where T is a fixed and finite time horizon. Suppose that the premium is paid continuously at the constant rate $c = (1+\theta)a$ with safety loading $\theta > 0$. According to Eq. (2.1), the surplus process $R(t)$ of an insurer is given by

$$\mathrm{d}R(t) = c\mathrm{d}t - \mathrm{d}C(t) = a\theta\mathrm{d}t + b\mathrm{d}W_0(t).$$

The insurer can purchase proportional reinsurance from the reinsurer, and $p(t)$ represents the proportion reinsured at time t. Then the surplus processes of the insurer and the reinsurer are

$$\mathrm{d}R_1(t) = (\theta - \eta p(t))a\mathrm{d}t + b(1-p(t))\mathrm{d}W_0(t),$$
$$\mathrm{d}R_2(t) = \eta p(t)a\mathrm{d}t + bp(t)\mathrm{d}W_0(t),$$

where $\eta > \theta$ represents the safety loading of the reinsurer. The net profit of the insurer is $(1+\theta)a-(1+\eta)p(t)a-(1-p(t))a$ and that of the reinsurer is $(1+\eta)p(t)a - p(t)a$. So the net profit conditions require that $0 \le p(t) \le \frac{\theta}{\eta} < 1$.

In this paper, the financial market consists of a risk-free asset (bond) and a risky asset (stock). The price of the risk-free asset $S_0(t)$ is

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t, \qquad S_0(0) = 1,$$

and the price of the risky asset $S(t)$ is described by the CEV model (see Xiao et al. [14])

$$\mathrm{d}S(t) = S(t)\left(\mu\mathrm{d}t + \sigma S(t)^{\beta}\mathrm{d}W(t)\right),$$

where r is the interest rate and μ is the expected instantaneous rate of the risky asset's return. $\sigma S(t)^{\beta}$ is the instantaneous volatility, and the elasticity β is a parameter which satisfies the general condition $\beta \le 0$. $W(t)$ is another standard Brownian motion defined on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, and $W(t)$ is independent of $W_0(t)$. As usual, we assume that $\mu > r$.