Journal of Computational and Applied Mathematics 255 (2014) 698–714

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Estimation in multivariate nonnormal distributions with stochastic variance function

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ARTICLE INFO

Article history:
Received 18 February 2013
Received in revised form 12 June 2013

Keywords:
Correlation coefficient
Least squares
Multivariate nonnormal distribution
Multivariate t-distribution
Modified maximum likelihood
Short-tailed distribution

ABSTRACT

In this paper the problem of estimation of location and scatter of multivariate nonnormal distributions is considered. Estimators are derived under a maximum likelihood setup by expressing the non-linear likelihood equations in the linear form. The resulting estimators are analytical expressions in terms of sample values and, hence, are easily computable and can also be manipulated analytically. These estimators are found to be remarkably more efficient and robust as compared to the least square estimators. They also provide more powerful tests in testing various relevant statistical hypotheses.

1. Introduction

Consider a q-variate random vector X having a location-scale type multivariate distribution with location vector Γ, and scale matrix Ω. Let, for any distinct pair $\left(X_j, X_k\right)$ $(2 \le k \le q,\ 1 \le j \le k-1)$ of random variables X_j and X_k in X, we have a bivariate distribution with location parameters μ_j and μ_k, scale parameters σ_j^2 and σ_k^2 and covariances $\sigma_{kj} = \rho_{kj}\sigma_k\sigma_j$ $\left(\sigma_{jk} = \sigma_{kj}\right)$, where ρ_{kj} $\left(=\rho_{jk}\right)$ is the Pearson correlation coefficient between random variables X_j and X_k. Let,

$$Z_j = \left(X_j - \mu_j\right)/\sigma_j \quad \text{and} \tag{1.1}$$

$$Z_k = \sqrt{(\nu_j + 1)/\nu_j}\sqrt{w_j}\left(X_k - \mu_{k\cdot j} - \theta_{kj}X_j\right)/\sigma_{k\cdot j}, \quad w_j = w_j\left(X_j\right) = 1/\left(1 + Z_j^2/\nu_j\right), \tag{1.2}$$

where $\mu_{k\cdot j} = \mu_k - \theta_{kj}\mu_j$, $\sigma_{k\cdot j}^2 = \sigma_k^2\left(1 - \rho_{kj}^2\right)$, and $\theta_{kj} = \left(\sigma_k/\sigma_j\right)\rho_{kj}$ are independently distributed as Student's t with ν_j and ν_k degrees of freedom, respectively. The conditional mean and variance of the variable X_k (condition to X_j) are

$$E(X_k \mid X_j = x_j) \propto \theta_{kj}x_j \quad \text{and} \quad V(X_k \mid X_j = x_j) \propto \sigma_{k\cdot j}^2/w_j(x_j), \tag{1.3}$$

respectively. Note that both are stochastic functions of X_j.

The problem is to find the maximum likelihood estimates (MLE) of the mean vector Γ and the scatter matrix Ω. However, the maximum likelihood equations have no explicit solution and solving them by iteration is problematic in various respects [1–4]. Therefore, we use the method of modified maximum likelihood estimation initially proposed by Tiku [5–8] and later on developed by Tiku, Islam and others [9–13]. The modified maximum likelihood estimates (MMLE) so obtained are in closed form and, hence, are easy to compute and can also be manipulated analytically. In fact, the MMLE are equivalent to MLE asymptotically [2,14–17]. Here we shall provide the MMLE in multivariate case following the approach used by Tiku et al. [12] for bivariate distributions.

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0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.06.032

2. Estimation

2.1. Modified maximum likelihood estimates

Consider a random sample (x_{ji}, x_{ki}) $(1 \le i \le n)$ taken from the bivariate distribution of random variable (X_j, X_k). The log likelihood function for the random sample can then be expressed as

$$\ln(L) \propto -n\ln\left(\sigma_j\right) - \frac{\left(\nu_j+1\right)}{2}\sum_{i=1}^{n}\ln\left(1+z_{ji}^2/\nu_j\right) - n\ln\left(\sigma_{k\cdot j}\right) - \frac{\left(\nu_k+1\right)}{2}\sum_{i=1}^{n}\ln\left(1+z_{ki}^2/\nu_k\right), \tag{2.1}$$

$$z_{ji} = \left(x_{ji}-\mu_j\right)/\sigma_j, \qquad z_{ki} = \sqrt{\nu_j^*}\sqrt{w_{ji}}\left(x_{ki}-\mu_{k\cdot j}-\theta_{kj}x_{ji}\right)/\sigma_{k\cdot j}, \quad \nu_j^* = (\nu_j+1)/\nu_j,\ w_{ji} = 1/(1+z_{ji}^2/\nu_j).$$

Here, the likelihood equations are expressed in terms of non-linear functions $g_l\left(z_{li}\right) = z_{li}/\left(1+z_{li}^2/\nu_l\right)$ $(1 \le i \le n;\ l = j, k)$ as

$$\mu_j : \sum_{i=1}^{n} g_j(z_{ji}) = 0, \tag{2.2}$$

$$\sigma_j : n - \nu_j^*\sum_{i=1}^{n} z_{ji}g_j(z_{ji}) = 0, \tag{2.3}$$

$$\mu_{k\cdot j} : \sum_{i=1}^{n} \sqrt{w_{ji}}g_k(z_{ki}) = 0, \tag{2.4}$$

$$\sigma_{k\cdot j} : n - \nu_k^*\sum_{i=1}^{n} z_{ki}g_k(z_{ki}) = 0, \tag{2.5}$$

$$\theta_{k\cdot j} : \sum_{i=1}^{n} \sqrt{w_{ji}}x_{ji}g_k(z_{ki}) = 0. \tag{2.6}$$

Solving these equations by iteration is fraught with difficulties as mentioned earlier. Instead, we first modify the likelihood equations by replacing z_{li} $(l = j, k)$ with corresponding ordered variates $z_{l(i)}$ $(l = j, k)$. The functions g_l $(l = j, k)$ are then replaced by their linear approximations $g_l\left(z_{l(i)}\right) \cong \alpha_{li} + \beta_{li}z_{l(i)}$ $(l = j, k)$ obtained from the first two terms of a Taylor series expansion about the ith quantiles t_{li} $(l = j, k)$ of Student's t-distributions with ν_l $(l = j, k)$ degrees of freedoms, respectively. This gives $\alpha_{li} = \left(2t_{li}^3/\nu_l\right)/\left(1+t_{li}^2/\nu_l\right)^2$, $\beta_{li} = \left(1-t_{li}^2/\nu_l\right)/\left(1+t_{li}^2/\nu_l\right)^2$ $(l = j, k)$. The values of t_{li} $(l = j, k)$ are given by $\frac{1}{\sqrt{\nu_l}B(1/2,\nu_l/2)}\int_{-\infty}^{t_{li}}\left(1+t^2/\nu_l\right)^{-(\nu_l+1)/2}dt = \frac{i}{n+1}$.

Denoting observation x_{ji} associated with $z_{j(i)}$ as $x_{j(i)}$ and solving the Eqs. (2.2) and (2.3) provides the MMLE of μ_j and σ_j $(1 \le j \le q)$ as follows:

$$\hat{\mu}_j = \bar{x}_{j(\cdot)}, \tag{2.7}$$

$$\hat{\sigma}_j = \left(B_j + \sqrt{B_j^2 + 4nC_j}\right)/2n, \tag{2.8}$$

where $\bar{x}_{j(\cdot)} = \sum_{i=1}^{n}\beta_{ji}x_{j(i)}/m_j$, $m_j = \sum_{i=1}^{n}\beta_{ji}$, $B_j = \nu_j^*\sum_{i=1}^{n}\alpha_{ji}x_{j(i)}$, $C_j = \nu_j^*\sum_{i=1}^{n}\beta_{ji}\left(x_{j(i)}-\bar{x}_{j(\cdot)}\right)^2$.

Now, take the observation (w_{ji}, x_{ji}, x_{ki}) associated with $z_{k(i)}$ and denote it as $(w_{j[i]}, x_{j[i]}, x_{k[i]})$ and call it concomitant of the observation (see [12]). The solution of Eqs. (2.4)–(2.6) provides the following MMLE for the correlation coefficient ρ_{kj} $(2 \le k \le q;\ 1 \le j \le k-1)$.

$$\hat{\rho}_{kj} = \left(\hat{\sigma}_j/\hat{\sigma}_k\right)\hat{\theta}_{kj}, \tag{2.9}$$

where

$$\hat{\theta}_{kj} = K_{kj} + D_{kj}\hat{\sigma}_{k\cdot j}, \quad \hat{\sigma}_{k\cdot j} = \left(B_{kj} + \sqrt{B_{kj}^2 + 4nC_{kj}}\right)/2n, \tag{2.10}$$

and

$$K_{kj} = \sum_{i=1}^{n}\beta_{ki}^*\left(x_{j[i]}-\bar{x}_{jk[\cdot]}\right)x_{k[i]} \Bigg/ \sum_{i=1}^{n}\beta_{ki}^*\left(x_{j[i]}-\bar{x}_{jk[\cdot]}\right)^2,$$

$$D_{kj} = \left(1/\sqrt{\nu_j^*}\right)\sum_{i=1}^{n}\alpha_{ki}^*\left(x_{j[i]}-\bar{x}_{jk[\cdot]}\right) \Bigg/ \sum_{i=1}^{n}\beta_{ki}^*\left(x_{j[i]}-\bar{x}_{jk[\cdot]}\right)^2,$$